Integer Quantum Hall Effect

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Chapter 1

Landau Levels Problem

1.1 Energy Spectrum

Here we take a brief look at the origin of Landau Levels.
Consider a spinless particle of charge $-e$ immersed in an external magnetic field. The Hamiltonian of the system can be written with the help of Peierl's rule for momentum substitution as:

$$H = \frac{(\mathbf{p} + e\mathbf{A})^2}{2m} \tag{1.1}$$

To simplify this we restrict the magnetic field to be only in the vertical axis ($\mathbf{B} = B\hat{z} = \nabla \times \mathbf{A}$). Define

$$\boldsymbol{\pi} = \mathbf{p} + e\mathbf{A} \tag{1.2}$$

Hence we obtain the following commutation relation:

$$[\pi_x, \pi_y] = -i\hbar eB \tag{1.3}$$

Next we define another couple of operators:

$$\begin{array}{rcl} a & = & \frac{1}{\sqrt{2e\hbar B}}(\pi_x - i\pi_y) \\ a^\dagger & = & \frac{1}{\sqrt{2e\hbar B}}(\pi_x + i\pi_y) \end{array} \tag{1.4}$$

This is very similar to the ladder operators in the harmonic oscillator problem. These operators even satisfy the commutation relation

$$[a, a^\dagger] = 1 \tag{1.5}$$

Using the ladder operators, we can construct a Hilbert space very similar to the ones in the quantum harmonic oscillator with a ground state defined by the annihilation when acted upon by the lowering operator and the subsequent higher states defined by the raising operator repeatedly acting on the ground state. We have now solved for the given Hamiltonian.
The energy of the nth state is given as:

$$E_n = \hbar\omega_B\left(n + \frac{1}{2}\right) \tag{1.6}$$

where $n = a^\dagger a$ is the number operator and $\omega_B = eB/m$ is the cyclotron frequency.
The energy spectrum before the introduction of external magnetic field was continuous. After the introduction of magnetic field into the system, the energy spectrum has been discretized! This is a drastic change to the system. The emergent discrete energy levels are called the Landau levels.

1.2 Landau Gauge

1.2.1 Eigenfunctions and Eigenspectrum

To solve for the wavefunctions of the Hamiltonian, we need to choose an appropriate gauge.

$$\mathbf{A} = xB\hat{y} \tag{1.7}$$

Even though the magnetic field we chose earlier was translation and rotation invariant in the xy plane, the vector potential breaks the translational symmetry in the $\hat{x}$ direction and the rotational symmetry (strips of equal potential along the $\hat{y}$ axis). As long as the physics we obtain from the calculations is invariant, we don't mind this hiccup.
Now we substitute this gauge into the Hamiltonian equation (1.1)

$$H = \frac{1}{2m}(p_x^2 + (p_y + eBx)^2) \tag{1.8}$$

We look for a solution by seperation of variables. Since we know that translation in the $\hat{y}$ direction is invariant, we can go one step further and guess that the function of y should resemble the plane wave solution.

$$\psi_k(x, y) = e^{iky} f_k(x) \tag{1.9}$$

We substitute this guess into the Landau gauge Hamiltonian and replace p_y with $\hbar k$. A few calculations and rearrangements later,

$$H_k\psi_k = \left[\frac{p_x^2}{2m} + \frac{1}{2}m\left(\frac{eB}{m}\right)^2\left(x + k\left(\frac{\hbar}{eB}\right)\right)^2\right]\psi_k \tag{1.10}$$

$$H_k = \frac{p_x^2}{2m} + \frac{1}{2}m\omega_B^2(x + kl_B^2)^2 \tag{1.11}$$

where $\omega_B = eB/m$ is the cyclotron frequency and $l_B = \sqrt{\hbar/eB}$ is the magnetic length - the characteristic length scale of quantum phenomena in a magnetic field of strength B (For 1 T magnetic field, $l_B \sim 2.5 \times 10^{-8}$ m).
This Hamiltonian looks exactly like a harmonic oscillator displaced from its centre of motion. Therefore the wavefunctions are Hermite polynomials in x and the energy spectrum is quantized with equal spacing between consecutive levels.

$$\begin{aligned} \psi_{n,k} &\sim e^{iky} H_n(x + kl_B^2) e^{-(x+kl_B^2)^2/2l_B^2} \\ E_n &= \hbar\omega_B\left(n + \tfrac{1}{2}\right) \end{aligned} \tag{1.12}$$

The wavefunctions look several strips laid down in the y axis which are exponentially localised at $-kl_B^2$.
We immediately get to see the degeneracy here. The wavefunctions depend on both n and k whereas the energy only depends on k.

1.2.2 Density of States

To find out the density of states, we restrict the xy plane to a finite area ($L_x \times L_y$).
In the y direction, the problem is exactly particle in a box. Hence, the density of states in the y direction is $2\pi/L_y$.
In the x direction, it is more difficult than that. With exponential localization at $-kl_B^2$ and the restriction within a 1D box of length L_x, we obtain:

$$\begin{aligned} 0 \leq \quad -kl_B^2 \quad &\leq L_x \\ -L_x/l_B^2 \leq \quad k \quad &\leq 0 \end{aligned} \tag{1.13}$$

Hence the total number of states is:

$$\begin{aligned} N &= \frac{L_y}{2\pi}\int_{-L_x/l_B^2}^{0} dk \\ &= \frac{L_x L_y}{2\pi l_B^2} \\ &= \frac{eBA}{2\pi\hbar} \end{aligned} \tag{1.14}$$

We can define the quantum of flux from the above relation.

$$N = \frac{BA}{\Phi_0} \tag{1.15}$$

where $\Phi_0 = 2\pi\hbar/e$ is the amount of flux that passes through an area $2\pi l_B^2$ Hence the density of states is:

$$n_B = \frac{eB}{2\pi\hbar} \tag{1.16}$$

This is exactly the density of states for a torus.

Important note: This does not imply that the flux through the entire system is quantized! The expression for density of states as a ratio of total flux through the system and the flux quantum is just a way of counting the density of states.

1.2.3 Hall Effect

To observe hall effect, we now add electric field in the $\hat{x}$ direction. The Hamiltonian is hence,

$$H = \frac{1}{2m}(p_x^2 + (p_y^2 + eBx)^2) + eEx \tag{1.17}$$

Complete the square to finally obtain:

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega_B^2\left(x + \frac{p_y}{eB} + \frac{mE}{eB^2}\right)^2 - \frac{1}{2}m\frac{E^2}{B^2} - \frac{p_y E}{2B} \tag{1.18}$$

whose eigenfunctions and eigenspectra are,

$$\begin{aligned} \psi(x,y) &= \psi_{n,k}\left(x + \tfrac{mE}{eB^2}, y\right) \\ E_{n,k} &= \hbar\omega_B\left(n + \tfrac{1}{2}\right) - eE\left(kl_B^2 + \tfrac{eE}{m\omega_B^2}\right) + \tfrac{1}{2}m\tfrac{E^2}{B^2} \end{aligned} \tag{1.19}$$

The degeneracy has been lifted. Now the energy spectrum is dependent on k as well. Hence, there is a drift in the y direction. Mutually perpendicular electric and magnetic field is producing a net drift in the $\mathbf{E}\times\mathbf{B}$ direction. This is nothing but the Hall effect!
The group velocity is given by

$$\begin{aligned} v_g &= \frac{1}{\hbar}\frac{\partial E_{n,k}}{\partial k} \\ &= -\frac{-k}{\hbar}l_B^2 \\ &= -\frac{E}{B} \end{aligned} \tag{1.20}$$

1.3 Symmetric Gauge

1.3.1 Introduction

Defined by

$$\begin{aligned} \mathbf{A} &= -\frac{1}{2}\mathbf{r}\times\mathbf{B} \\ &= -\frac{yB}{2}\hat{x} + \frac{xB}{2}\hat{y} \end{aligned} \tag{1.21}$$

Here translation symmetry is broken in both x and y direction but rotation symmetry is preserved. Hence angular momentum about the origin is a good quantum number.

Introduce $\tilde{\boldsymbol{\pi}} = \mathbf{p} - e\mathbf{A}$
We then have the following results:
$[\tilde{\pi}_x, \tilde{\pi}_y] = ie\hbar B$
$[\pi_x, \tilde{\pi}_x] = 2ie\hbar \frac{\partial A_x}{\partial x}$
$[\pi_y, \tilde{\pi}_y] = 2ie\hbar \frac{\partial A_y}{\partial y}$
$[\pi_x, \tilde{\pi}_y] = [\pi_y, \tilde{\pi}_x] = ie\hbar\left(\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y}\right)$
In the symmetric gauge, except for the first commutator relation, everything else goes to zero.
Now we define two new ladder operators with $\tilde{\boldsymbol{\pi}}$

$$\begin{aligned} b &= \tfrac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x + i\tilde{\pi}_y) \\ b^\dagger &= \tfrac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x - i\tilde{\pi}_y) \end{aligned} \tag{1.22}$$

They satisfy the commutation relation $[b, b^\dagger] = 1$. The second set of ladder operators provide degeneracy to the Landau levels.
Define the ground state with respect to the first and second ladder operator as: $|0,0\rangle$

$$a\,|0,0\rangle = b\,|0,0\rangle = 0 \tag{1.23}$$

The general state $|n,m\rangle$ and energy are given by:

$$\begin{aligned} |n,m\rangle &= \tfrac{(a^\dagger)^n (b^\dagger)^m}{\sqrt{n!m!}}\,|0,0\rangle \\ E_n &= \hbar\omega_B\left(n + \tfrac{1}{2}\right) \end{aligned} \tag{1.24}$$

1.3.2 Lowest Landau Level

The lowest state is representated by: $|0,m\rangle$. To obtain the ground state wavefunction, we need a differential equation to solve. We know that $a\,|0,m\rangle = 0$. We substitute the functional form of p and A and we obtain,

$$a = \frac{1}{\sqrt{2e\hbar B}}\left[-i\hbar\left(\partial_x - i\partial_y\right) + \frac{eB}{2}(-y - ix)\right] \tag{1.25}$$

We now move to the complex plane: $z = x - iy$ and $\bar{z} = x + iy$. This way of definition is chosen so that the wavefunction is holomorphic for positive magnetic field. Define new operators:

$$\begin{aligned} \partial &= \tfrac{1}{2}(\partial_x + i\partial_y) \\ \bar{\partial} &= \tfrac{1}{2}(\partial_x - i\partial_y) \end{aligned} \tag{1.26}$$

$\partial z = \bar{\partial}\bar{z} = 1$. We now express the ladder operators using the new operators.

$$\begin{aligned} a &= -i\sqrt{2}\left(l_B\bar{\partial} + \tfrac{z}{4l_B}\right) \\ a^\dagger &= i\sqrt{2}\left(l_B\bar{\partial} - \tfrac{\bar{z}}{4l_B}\right) \end{aligned} \tag{1.27}$$

The differential equation is for the lowest Landau level is:

$$a\psi_{LLL}(z,\bar{z}) = 0 \tag{1.28}$$

The solution is:

$$\psi_{LLL} = f(z) \exp\left(- \frac{|z|^2}{4l_B^2} \right) \tag{1.29}$$

for any holomorphic function $f(z)$. Similarly, we express the second set of ladder operators.

$$\begin{aligned} b &= -i\sqrt{2}\left(l_B \partial + \tfrac{\bar{z}}{4l_B} \right) \\ b^\dagger &= i\sqrt{2}\left(l_B \bar{\partial} - \tfrac{z}{4l_B} \right) \end{aligned} \tag{1.30}$$

To get the functional form of the true ground state $|0,0\rangle$, we need to solve for $a\psi_{LLL,m=0} = 0$ and $b\psi_{LLL,m=0} = 0$. The result is:

$$\psi_{LLL,m=0} \sim \exp\left(- \frac{|z|^2}{4l_B^2} \right) \tag{1.31}$$

which are eigenstates of angular momentum operator. The higher Landau levels can be obtained by repeatedly acting the $a^\dagger$ operator upon the LLL function.

Chapter 2

2DEG

2.1 Drude Model

For a 2D electron gas in mutually perpendicular electric and magnetic field (as described before), we apply the Drude model. The equation of motion is given by:

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - e\mathbf{v}\times\mathbf{B} - \frac{m\mathbf{v}}{\tau} \tag{2.1}$$

The last term is a linear friction term. It is a catch all term that encompasses all kinds of interactions that slow down the electron motion - electron-electron interaction, collisions with impurities etc. τ is the average relaxation time period.

We are interested in the equilibrium solution (i.e. $d\mathbf{v}/dt$). Hence we get:

$$\mathbf{v} + \frac{e\tau}{m}\mathbf{v}\times\mathbf{B} = -\frac{e\tau}{m}\mathbf{E} \tag{2.2}$$

Current density is given by: $\mathbf{J} = -ne\mathbf{v}$. Substituting this back into the previous equation and seperating different components the vector, we get:

$$\begin{aligned} J_x + \left(\tfrac{eB}{m}\right)\tau J_y &= \tfrac{ne^2\tau}{m}E_x \\ -\left(\tfrac{eB}{m}\right)J_x + \tau J_y &= \tfrac{ne^2\tau}{m}E_y \end{aligned} \tag{2.3}$$

Put $\sigma_{DC} = ne^2\tau/m$ and $\omega_B = eB/m$ and convert the above equations into matrix form.

$$\mathbf{E} = \hat{\rho}\mathbf{J} \tag{2.4}$$

where $\hat{\rho}$ is the resistivity tensor which comes out to be:

$$\hat{\rho} = \frac{1}{\sigma_{DC}}\begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{pmatrix} \tag{2.5}$$

The conductivity tensor is the inverse of resistivity tensor.

$$\hat{\sigma} = \frac{\sigma_{DC}}{1+\omega_B^2\tau^2}\begin{pmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{pmatrix} \tag{2.6}$$

The conductivity tensor is defined as:

$$\hat{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix} \tag{2.7}$$

where σ_{xx} is the longitudinal conductivity and σ_{xy} is the Hall conductivity. Similarly, ρ_{xx} and ρ_{xy} are the longitudinal and Hall resistivities. The resistivity tensor elements in terms of the conductivity tensor elements are:

$$\begin{aligned} \rho_{xx} &= \frac{\sigma_{xx}}{\sigma_{xx}^2+\sigma_{xy}^2} \\ \rho_{xy} &= \frac{\sigma_{xy}}{\sigma_{xx}^2+\sigma_{xy}^2} \end{aligned} \tag{2.8}$$

An incredible thing we notice here is that when the longitudinal conductivity (σ_{xx}) goes to zero, the longitudinal resistivity (ρ_{xx}) goes to zero as well! This means that a 2D material with zero resistivity will not be a perfect conductor and vice versa! All the notions of conductors and resistors for a bulk material need to be modified for a 2D material. Here we see that the conductivity of a material not only depends on its electronic configuration but also on the physical dimensions of the material. The reason for zero conductivity for 2DEG is that the fermi energy lies in the gap between the highest occupied energy level and the lowest unoccupied energy level, making $\sigma_{xx} = 0$ (this is when the edge current takes over). Also, the allowable phase space for scattering states vanishes because of this, making $\rho_{xx} = 0$.
Let us now consider the case where there is no mechanism for relaxation for the electron. We move to the reference frame where the electron is experiencing no force i.e. Lorentz force is zero:

$$\mathbf{v} \times \mathbf{B} = -\mathbf{E} \tag{2.9}$$

We recognise here that the velocity of the electron is in the x direction and the magnetic field is in the z direction, hence we obtain the relations:

$$\begin{aligned} E_y &= vB \\ E_x &= 0 \end{aligned} \tag{2.10}$$

We know that $\mathbf{E} = \hat{\rho}\mathbf{J}$ and that $\mathbf{J} = -ne\mathbf{v}$. We substitute to find:

$$\begin{pmatrix} 0 \\ vB \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix} \begin{pmatrix} -nev \\ 0 \end{pmatrix} \tag{2.11}$$

This leads to the following expressions for the matrix elements of resistivity tensor:

$$\begin{aligned} \rho_{xx} &= 0 \\ \rho_{xy} &= \frac{B}{ne} \end{aligned} \tag{2.12}$$

This is another remarkable conclusion! This set of equations tell us that even when the longitudinal resistivity is zero, the Hall conductivity need not be zero and it increases with the strength of the magnetic field! The elements of the conductivity tensor are:

$$\begin{aligned} \sigma_{xx} &= 0 \\ \sigma_{xy} &= \frac{ne}{B} \end{aligned} \tag{2.13}$$

Hence, in the longitudinal direction, we have zero resistivity and conductivity for the 2DEG system, but the hall resistivity and conductivity are non-zero.

2.2 Origins of Quantization of Hall Conductivity

2.2.1 Lorentz Invariance Method

When the electrons do not interact, we can consider them as single particles streaming along. This flashes us back to the Landau levels in a finite rectangular box. We stay in the same Lorentz invariant frame. The

total number of states is given by equation (1.14). If each state is considered to have one electron (spinless particles) and there are a total of ν filled Landau levels each of which have N number of k states, then the current density flowing in the x direction is:

$$\begin{aligned} j &= \frac{e\nu N v}{L_x L_y} \\ j_y &= \frac{e\nu N E}{B L_x L_y} \\ j_x &= 0 \end{aligned} \tag{2.14}$$

Substitute into $\mathbf{j} = \hat{\sigma}\mathbf{E}$ and use equation (1.15) to obtain the Hall conductivity:

$$\begin{aligned} \sigma_{xy} &= \frac{eN\nu}{BL_xL_y} \\ &= \frac{eN\nu}{N2\pi\hbar/e} \\ &= \nu\frac{e^2}{h} \end{aligned} \tag{2.15}$$

Now we need to define the number of occupied Landau levels. Equation (1.16) tells us the density of states (n_B). Then $1/n_B$ is the area occupied by each state in a level. Then, the number of occupied Landau levels is then given by:

$$\nu = \frac{n_e}{n_B} \tag{2.16}$$

where n_e is the number density of electrons in the system. In the case of integer quantum hall effect, the number of occupied Landau levels is an integer. Hence the Hall conductivity of a Landau level must be integer multiples of e^2/h.

2.2.2 Quantum Mechanical Method

As we saw in the previous section, for completely occupied Landau levels, the Hall conductivity is an integer (the Landau level itself) multiple of e^2/h. What if the last Landau level is partially filled? To explain that, we do a quantum mechanical calculation of hall conductivity through which we will be able to see the answer.

We know that the velocity of a single electron is given by:

$$m\dot{\mathbf{x}} = \mathbf{p} + e\mathbf{A} \tag{2.17}$$

and the current is given by $\mathbf{I} = -e\dot{\mathbf{x}}$. Hence the total current is given by the taking the quantum mechanical average over all Landau levels and their k states:

$$\mathbf{I} = -\frac{e}{m}\sum_{n=1}^{\nu}\sum_{k}\langle\psi_{n,k}|\,\mathbf{p} + e\mathbf{A}\,|\psi_{n,k}\rangle \tag{2.18}$$

where ν is the number of filled Landau levels. We go with the Landau gauge with an electric field in the x direction. Now we calculate the currents in x and y direction:

$$I_x = -\frac{e}{m}\sum_{n=1}^{\nu}\sum_{k}\langle\psi_{n,k}|p_x|\psi_{n,k}\rangle = 0 \tag{2.19}$$

Since the wavefunctions are same as harmonic oscillator eigenstates, the momentum in the x direction has zero expectation value at all states.

$$\begin{aligned} I_y &= -\tfrac{e}{m}\textstyle\sum_{n=1}^{\nu}\sum_{k}\langle\psi_{n,k}|\,p_y + exB\,|\psi_{n,k}\rangle \\ &= -\tfrac{e}{m}\textstyle\sum_{n=1}^{\nu}\sum_{k}\langle\psi_{n,k}|\,\hbar k + exB\,|\psi_{n,k}\rangle \end{aligned} \tag{2.20}$$

We know from equation (1.19) that the position of x is shifted by an amount $-kl_B^2 - eE/m\omega_B^2$ which is the expectation value of x we will get from here. Substituting that, we finally get:

$$\begin{aligned} I_y &= -\tfrac{e}{m}\textstyle\sum_{n=1}^{\nu}\sum_k eB\left(\frac{mE}{eB^2}\right) \\ &= e\nu\textstyle\sum_k \frac{E}{B} \\ &= e\nu\frac{BA}{\Phi_0}\frac{E}{B} \\ &= e^2\nu EA/h \end{aligned} \tag{2.21}$$

where A, here, is the area. Now if we subsititute this into the current density and solve for the microscopic version of Ohm's law, we get the conductivity tensor with values given by:

$$\begin{aligned} \sigma_{xx} &= 0 \\ \sigma_{xy} &= \nu\tfrac{e^2}{h} \end{aligned} \tag{2.22}$$

Nothing new to see here. We started off with completely filled Landau levels and performed quantum mechanical average calculations for the current in the system and found out that longitudinal conductivity was zero while hall conductivity was quantized. This means that if we go from a state of ν filled levels to $\nu + 1$ filled levels, the hall conductivity will take a leap. But to go get there, we need to fill all the k states of the $(\nu + 1)$th level. What happens if we fill it partially?

Let us take another look at the calucations above. The value of I_x is clearly not going to change because the average of p_x is identically zero for all k states. Whereas I_y will pick up an extra term eE/B for the extra k' states of $\nu + 1$. Hence the current density in the y direction turns out to be:

$$\begin{aligned} J_y &= \tfrac{e^2\nu}{h}E + \textstyle\sum_{k'}\frac{eE}{BA} \\ &= \tfrac{e^2E}{h}\left(\nu + \frac{k'}{N}\right) \end{aligned} \tag{2.23}$$

where $N = BA/\Phi_0$ is the total number of k states within a Landau level. Therefore, the hall conductivity is given by:

$$\sigma_{xy} = \frac{e^2}{h}\left(\nu + \frac{k'}{N}\right) \tag{2.24}$$

Since the density of k states in a Landau level need not be an integer as mentioned in a note above, the hall conductivity given here is not quantized. In fact, it is a continuous, monotonically increasing function.

Chapter 3

Interlude: Comments

3.1 To Conduct or to Insulate?

If the resisitivity is zero in a normal metal, we can call it a perfect conductor with infinite conductivity and vice versa for perfect insulator. In the case of a 2DEG, both of them go to zero simultaneously. What must one call this then?
Let us start from the basics: there is a system of electrons constrained in 2D and a uniform magnetic field is pointing in the perpendicular direction. For Ohm's law in a conducting wire, current flows along the electric field. In this current instance however, we see that if an electric field is applied in the x direction, the response is seen in the y direction - perpendicular to both the electric and magnetic fields. Hence this system cannot be classified as a conductor or an insulator in the bulk matter sense of the word.

3.2 Disorder

All of the arguments in the preceding sections hinged on a very strong assumption that the system is translationally invariant in the y direction. That is how we got everything from the wavefunctions, the spectra with Landau levels, density of states and conductivity and resistivity tensors. In real materials, it is seen that instead of the taken at each k filling of a Landau level leading to a step increase in hall conductivity, it is seen that the hall conductivity value stays at the same value until the next Landau level is completely filled where it takes a big step increase to the predicted value. If the applied magnetic field is decreased from $B = n_e\Phi_0/\nu$ to a value above $B = n_e\Phi_0/(\nu+1)$, we expect the conductivity to appropriately rise. But it turns out that this is not the case. The conductivities remain unchanged until the next level is reached. What is the reason for this? Disorder.

By putting in disorder in the system, we make the system no longer translationally invariant. The Landau levels broaden into a Lorentzian distribution as a result with the middle of the distribution being the unperturbed (extended) state. We want to work in the regime where the Lorentzian distributions are distinguishable from one another (just a pinch of disorder, not too much).

This is where the conductivity of the material comes in. Existance of extended states implies that transport of an electron between the two ends of the material is possible. Once localization sets in, the electrons get trapped within the localized states and the conductivity value takes a hit. Only the extended states contribute to the conductivity of a material. Let us apply this knowledge to the Landau landau levels of electrons. Intially, let the Fermi energy of the system be in middle of one of the broadened Landau levels (extended state). Suppose we keep the number density of electrons (n_e) fixed and decrease the magnetic field B. A look back at equation (2.16) tells us that the number of electrons in each level would decrease, thereby

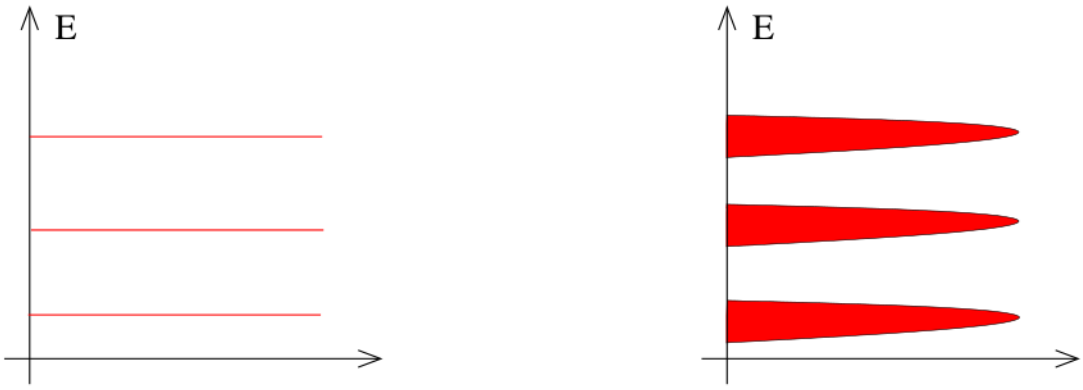

Figure 3.1: Left: Density of states without disorder. Right: Density of states with disorder

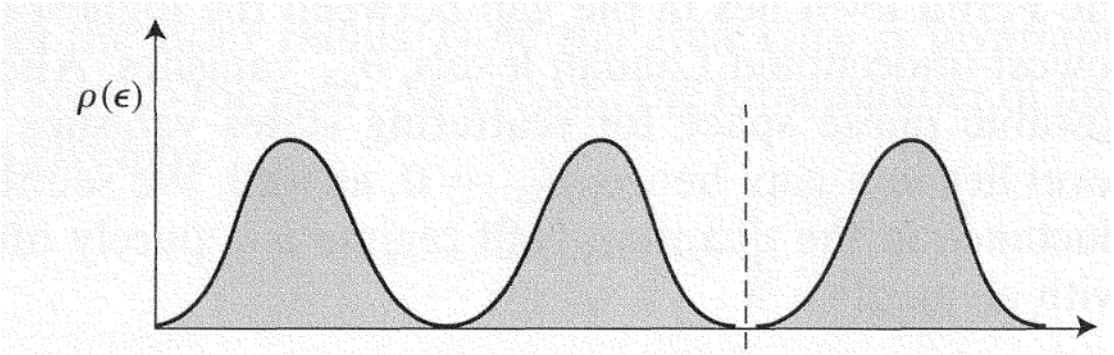

Figure 3.2: Density of states with disorder. The dotted line signified the Fermi energy of the system.

increasing the fermi energy. Hence the fermi energy moves into the regime of the Lorentzian distribution where localized states exist. This implies that electrons will go into the localized states that arise due to the disorder in the system (figure 3.2). As we keep adding electrons, the fermi energy keeps moving along the localized states until it encouters the next extended state. Occupation of localized states does not affect the conductivity of the system, only extended states do. Hence when the localized states are being filled, the hall conductivity stays at $\sigma_{xy} = \nu e^2/h$ until the next Landau level is encountered, which forms the plateaux. The density of states looks as shown in figure (3.3). This assumption is very important for the starting point of Laughlin's quantization of Hall conductivity. The density of states for a disordered 2D system, must have a band of extended states in the middle followed by localized states or no states.

Since the electrons are trapped in localized states, the hall resistivity should increase as the magnetic field is decreased when in between the two Landau levels, hence deviating from quantized behaviour. But this is not the case. As more electrons are trapped in localized states, the current carried by the extended states increases such that it is exactly compensated. Hence the hall resistivity also follows quantized behaviour.

We have explained why the plateaux form and have phenomenologically obtained the quantum Hall conductivity values but we still haven't explained why the hall conductivity takes specific quantized values. For this we take a look at Laughlin's calculations.

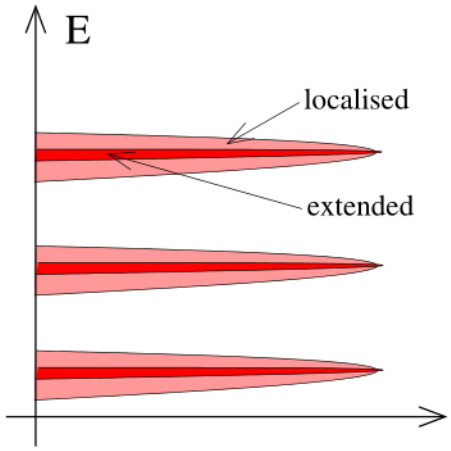

Figure 3.3: The states away from the centre are localized but are extended at the centre of the band

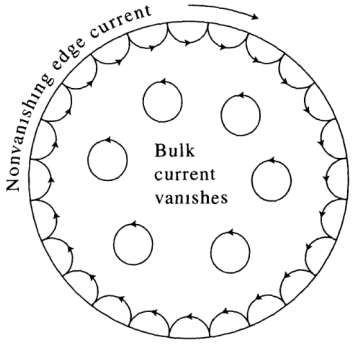

Figure 3.4: Cyclotron orbits and edge currents in a quantum Hall system. Their chiralities are opposite.

3.3 Edge States

In a quantum Hall system, electrons to follow a cyclotron orbit due to Lorentz force. The direction of the orbit is determined by the directions of electric and magnetic field. In the interior of the bulk material, the current produced by the first half of an electron orbit is exactly cancelled by the second half of the orbit of a neighbouring electron. Hence there is no net current within the material. At the edge of the material, however, where the electron is unable to complete a full circular orbit without encountering finite potential at the boundary of the material, we see that the electron gets reflected off the edge and tries to continue the cyclotron motion. This skipping motion at the edge of the material leads to an edge current in the direction opposite to the direction of cyclotron motion within the bulk. This argument is valid when the magnetic length is lesser than the wavelength of the electron. Current in the system is only carried at the edge.

Chapter 4

Laughlin's Work

Laughlin took a system as shown:

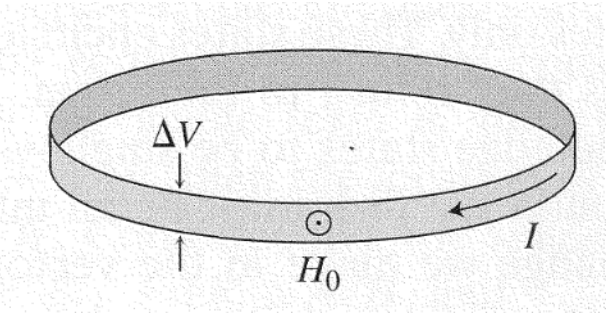

Figure 4.1: Laughlin's system: A quantum Hall ribbon with magnetic field everywhere perpendicular to the surface. Circumference L, width W, Hall voltage $\Delta V = E_0 L$

We take y axis to run along the circumference of the ribbon and the x axis along the width of the ribbon. The magnetic field is pointing in perpendicular to the surface of the ribbon. This system is the same as the rectangular gas we took before except with periodic boundary conditions. This is an important caveat. If the y boundary is periodic, we expect the wavefunction of extended states to be phase coherent after one trip around the ribbon.

We consider our system to be unperturbed for small changes in the vector potential. We introduce a change in vector potential by introducing a vanishingly thin infinitely long solenoid which has zero magnetic field but non-zero vector potential outside the solenoid. We place the solenoid in the system such that the axis of the solenoid is parallel to the axis of the ribbon. This is the Aharonov-Bohm flux. By introducting the Aharonov-Bohm flux into the system, we have punctured a hole in the topology of the system, which would not have been possible with the rectangular electron gas. There now exists a pole with infinite magnetic field. The additional vector potential/flux contributes to the phase of the wavefunction as: $e^{ieAy/\hbar}$.

For localized states, the phase of the wavefunction changes as indicated above with a continuous gauge trasnformation. That can be set to zero by a suitable local gauge transformation.

For the extended states, however, due to the change in topology of the system, not all gauge transformations will be allowed. Extended states coherently stretch all the way around the ribbon. Hence we have to fulfill the PBC:

$$\psi_{n,k}(x, y + L)e^{ieA(y+L)/\hbar} = \psi_{n,k}(x, y)e^{ieAy/\hbar} \tag{4.1}$$

This implies that only certain values of A are allowed for gauge transformation to be possible:

$$A = n\frac{h}{eL} \tag{4.2}$$

Here we see that the value of vector potential and hence the additional flux entering the system must be quantized.

The energy of the system is given by $\langle\psi_{n,k}| H |\psi_{n,k}\rangle$. According to Hellman-Feynman theorem,

$$\frac{\partial E}{\partial \lambda} = \langle\psi_{n,k}| \frac{\partial H}{\partial \lambda} |\psi_{n,k}\rangle \tag{4.3}$$

for some parameter in the Hamiltonian λ. We consider the Hamiltonian in equation (1.1) and vary its energy with respect to the vector potential.

$$\begin{array}{rcl} \frac{\partial \epsilon_{n,k}}{\partial A} & = & \frac{e}{m}\langle\psi_{n,k}| \mathbf{p} + e\mathbf{A} |\psi_{n,k}\rangle \\ & = & LI_{n,k} \end{array} \tag{4.4}$$

where $I_{n,k}$ is the angular current in the n,k state. Hence we obtain the following expression for current:

$$I_{n,k} = \frac{1}{L}\frac{\partial \epsilon_{n,k}}{\partial A} \tag{4.5}$$

We work with the Landau gauge. The Hamiltonian scales linearly with respect to x_k when an electric field is added (eE_0x_k) where x_k is the location of the centre of the states in the Landau level. Therefore if we modify the gauge term as $\mathbf{A} \to (Bx + \Delta A)\hat{y}$, then the centre shifts in value $x_k \to x_k + \Delta A/B$ and the energy shifts as $\epsilon_{n,k} \to \epsilon_{n,k} - eE_0\Delta A/B$. We know this already from the calculations done above. Hence we obtain the derivative as:

$$\frac{\partial \epsilon_{n,k}}{\partial A} = -\frac{eE_0}{B} \tag{4.6}$$

For current to pass in the system, we want the electrons in some Landau level localized at some point to move to its neighbouring centre and the neighbour to its neighbour and so on. This will create a chain of electrons jumping centres which signifies a net transfer of one electron per Landau level from one edge to the other. For this, the change in the position of the centre of wavefunction due to the change in vector potential must be equal to the distance between two neighbourhood centres m and $m+1$.

$$\begin{array}{rcl} \Delta x_k & = & x_k^{m+1} - x_k^m \\ & = & l_B^2(k_{m+1} - k_m) \\ & = & (2\pi/L)l_B^2 = h/eBL \end{array} \tag{4.7}$$

Therefore for the transfer for one electron from one end to the other per Landau level, the change in vector potential must be:

$$\Delta A = \frac{h}{eL} \tag{4.8}$$

The resulting flux produced due to the change in vector potential as above can be called as a single flux quantum. If there are n electrons transferred, we multiply the right hand side of the equation by n. If a n flux quanta are threaded through the ribbon, it leads to the transfer of n electrons per Landau level from one edge to the other. If the flux quanta are threaded in the opposite chirality, then the electrons are transferred in the opposite direction. Here we see that the winding number of vector potential/flux is is equal to the number of electrons per Landau level transferred from one edge to the other (and hence equal to the hall current and conductivity - up to a constant multiplication). This is why the explanation is topolgical in nature.

$$\begin{array}{rcl} I & = & \frac{1}{L}\sum I_{n,k} \\ & = & -\frac{en_eE_0}{L_B} = \frac{e\nu n_B V_H}{B} \\ & = & -\frac{\nu e^2 V_H}{h} \end{array} \tag{4.9}$$

where $V_H = E_0 W$ is the hall voltage. Since the current is quantized, the current density is quantized which implies that the conducivity and resistivity are quantized.

This argument gives a complete idea of why plateaux exist. If the Aharonov-Bohm flux though the ribbon is some fraction, then the extended state cannot exist because gauge invariance is broken. Hence we only have localized states within the ribbon which do not contribute to the hall conductivity, hence plateaux. Whereas if the value is cranked up to an integer, then we obtain extended states, with a jump in hall conductivity.

A very important fact to note here is that we consider the external vector potential to be added to the system **adiabatically**. This assumption is needed because we do not want this external flux to modify the wavefunction of the original system.

Chapter 5

Halperin's Work

Halperin considered a noninteracting system in a geometry show in the figure below.

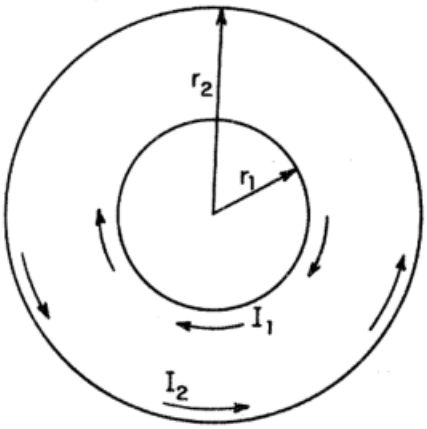

Figure 5.1: Annular film. Inner radius r_1 and outer radius r_2. Uniform magnetic field B_0 perpendicular to plane of annulus. Additional flux in the region $r < r_1$. Direction of currents in the boundaries shown with arrows.

The additional flux within the hole of the annulus is adjustable and does not affect the magnetic field in the annular region and is increased adiabatically. Also, there is no external electric field and the dimensions of the annulus are much larger than the cyclotron radius for electrons in the magnetic field (r_c). The gauge is chosen as:

$$A = \frac{1}{2}B_0 r + \frac{\Phi}{2\pi r} \tag{5.1}$$

The Hamiltonian of the system is of the form given by equation (1.1). Again, due to the periodicity in the θ direction, we have a plane wave solution with integer valued canonical momentum for θ whereas a displaced quantum harmonic oscillator solution with respect to r. The wavefunction is given by:

$$\begin{aligned} \psi_{m,\nu} &= e^{im\theta} r^m e^{-r^2/4l_B^2} \\ E_\nu &= \hbar\omega_c\left(\nu + \tfrac{1}{2}\right) \end{aligned} \tag{5.2}$$

where $\omega_c = eB_0/m$ is the cyclotron frequency and $r_m \approx \sqrt{2ml_B^2}$ is the point of localization of r. It also satisfies the following equation for effective flux through the system.

$$B_0(\pi r_m^2) = m\Phi_0 - \Phi \tag{5.3}$$

where $\Phi_0 = h/e$ is the flux quantum. We inherently assume here that $r_1 < r_m < r_2$ and that $r_m - r_1 < r_c$ and $r_2 - r_m < r_c$. Only if these are true, the wavefunction is a valid solution to the Hamiltonian.

We now calculate the current along the circular path as we have done before: finding the average velocity of electrons and adding them up over all states.

$$\begin{aligned} I_{m,\nu} &= \tfrac{e}{m}\int_0^\infty dr|\psi_{m,\nu}|^2\left(\tfrac{L_\theta}{r} + eA(r)\right) \\ &= \tfrac{e}{m}\int_0^\infty dr|\psi_{m,\nu}|^2\left(\tfrac{m\hbar}{r} + eA(r)\right) \end{aligned} \tag{5.4}$$

We know that the electron density is symmetric about r_m and we assume it decays rapidly as we move futher away from r_m. Under this approximation, the current becomes:

$$I_{m,\nu} = \frac{e^2B_0}{m}\int_0^\infty dr|\psi_{m,\nu}|^2(r_m - r) \tag{5.5}$$

This net current vanishes because the probability density is a symmetric function about r_m.

Near the outer edge of the annulus, the wavefunction is written as:

$$\psi_{m,\nu} = e^{im\theta}g_\nu(r - r_m, r_2 - r_m) \tag{5.6}$$

where $g_\nu(x, s)$ is a wavefunction defined such that the function vanishes at $x \to s$ and $x \to -\infty$ and satisfies the eigenvalue equation:

$$\frac{(p_x^2 + e^2A_x^2)}{2m}g_\nu = Eg_\nu \tag{5.7}$$

The energy curve will look as shown. At the middle of the annulus, the energy is equal to $\hbar\omega_c(n + 1/2)$ and there is no variation. As we approach the edges, we see that the energy increases many folds.

The current is no longer zero because the density function is no longer symmetric about r_m. Current is given by:

$$I_{m,\nu} = -\frac{\partial E_{m,\nu}}{\partial\Phi} = \frac{e}{h}\frac{\partial E_{m,\nu}}{\partial m} \tag{5.8}$$

Hence the current within the bulk of the annular region is zero whereas at the edges, it is positive (counter-clockwise) for $r_m \approx r_2$ and negative (clockwise) for $r_m \approx r_1$.

Suppose we increase the external flux slowly from 0 to Φ_0 while the fermi surface exists between two Landau levels. Adiabatic theorem tells us that the electron occupations should not change. But what changes is the equilibrium position $r_m \to r_{m+1}$. For a particular filled Landau level, this leads to every electron localizing at a neighbourhood site which effectively transfers an electron from one edge to the other. If there are ν filled Landau levels, then ν electrons are transported. If the edge states are at different fermi energies the current is given by:

$$I_{m,\nu} = \nu e\Delta E/h \tag{5.9}$$

Now let us add disorder in the system. A weak random potential $|V(r)| \ll \hbar\omega_c$. Again we consider that the fermi level lies between the zeroth and the first Landau levels. Write the eigenstate of the new system at r_2 to be a superposition of eigenstates of the nonrandom system:

$$\psi(r) = \sum_{m,\nu} c_{m,\nu}\psi_{m,\nu}(r) \tag{5.10}$$

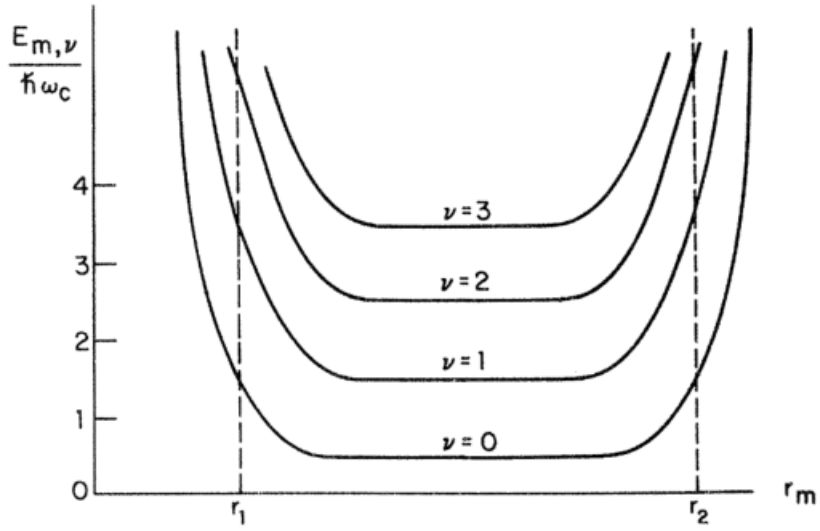

Figure 5.2: Energy levels as a function of distance in units of $\hbar\omega_c$

The factor $c_{m,\nu}$ will be exponentially small for $|r_2 - r_m| \gg r_c$, smaller by orfer $V/\hbar\omega_c$ for $\nu \geq 1$ and relatively large for $\nu = 0$ and r_m close to r_2.

The angular current is given by density calcualtion again:

$$< I >= \sum_{m\nu\nu'} c^*_{m\nu} c_{m\nu'} I_{m\nu\nu'} \tag{5.11}$$

where

$$I_{m\nu\nu'} = \frac{e}{2\pi m} \int\int dr d\theta \psi^*_{m\nu} \psi_{m\nu'} \left(\frac{m\hbar}{r} - eA(r) \right) \tag{5.12}$$

when $\nu' = \nu$ the above is nothing but $I_{m,\nu}$. The diagonal contribution of the matrix above is not completely eliminated by the non-diagonal part if we consider the approximations for the constants as mentioned (and the random potential is small compared to the cyclotron energy). Hence the average angular current is nonzero. It is also independent of the angle θ, hence we can conclude that the value is uniform throughout the annular region. Even if there are isolated regions where the potential is much stronger compared to the cyclotron energy, the current carrying edge states will find a way to go around the localized regions formed by the stong potentials. If the average potential is sufficiently strong, then the scattering rate will be large compared to the cyclotron frequency and the arguments given will break down.

Chapter 6

Variation from quantized nature in experiments

For all the calculations that we have done, we never included the role of temperature into the system. We only worked in the case where the system is in absolute zero temperature. It is impossible to achieve absolute zero in lab conditions, however, one can go quite close to abolute zero. Unfortunately, even at temperatures of order lesser than 1K, the exact quantum nature of hall conductivity breaks down due to the nonzero longitudinal resistivity. In the disordered system, the nonzero temperature aids hopping of electrons from one localized state to another. This is know =n as variable-range hopping. If the temperature is raised even further upto order of 1K, the Fermi Dirac statistics of electrons allows occupation of extended states and hence futher increases the value of longitudinal resistivity.

Chapter 7

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