Quantum Weak Measurement and its Implementation in Optical Systems

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Chapter 1

Introduction

1.1 The Essence of Weak Measurement

In quantum physics, measurement is an operation on a wavefunction (that can be expressed as a superposition of the operator eigenfunctions) that completely changes the state of a wavefunction by making it collapse into one of the eigenstates of the operator. The eigenvalue of the final state is defined as the output of that measurement. This general notion of measurement can be termed as "strong" measurement since the memory of the previous mixed state wavefunction is completely lost once the measurement process is completed.

If, somehow, we can perform a measurement such that the mixed state of the wavefunction that the measurement is being performed upon, is not completely lost, then such a measurement process is called "weak" measurement. To illustrate, let us look at the Stern-Gerlach experiment.

1.2 Stern-Gerlach Experiment

A well known experiment in quantum mechanics is the Stern-Gerlach experiment. A collimated beam of silver atoms are subjected to an inhomogeneous magnetic field produced by a pair of magnetic poles. This setup is shown in the figure below. What happens next is the essence of quantum mechanics. The silver atom has one electron in its valence shell. All the rest of the electrons can be considered to form a spherically symmetric cloud around the nucleus. If we ignore the spin of the nucleus and consider that only electrons have spin, the magnetic field will deflect the atom due to the effective spin of the valence electron silver atom has. Classically, the stream of silver atoms can have their spins oriented randomly in all directions. Hence, the deflection due to the magnetic field must form a straight line parallel to the field gradient. The intensity at each point begin proportional to the cosine of the angle between the magnetic field and the spin of the atom.

In reality, there are only two dots that appear on the screen corresponding to two orientations of spin. One corresponding to spin "up" and another corresponding to spin "down", both with equal intensities. This could be done in any axis direction and we would only get two spots on the screen. This empirically proves that the electron only has two value of spin "up" and "down", the orientation of which, depends on the axis we choose to measure it from.

Sequential Stern-Gerlach experiments reveal that the "memory" of the atom being z spin up or down, vanishes once it is passed through another Stern-Gerlach apparatus in a perpendicular orientation.

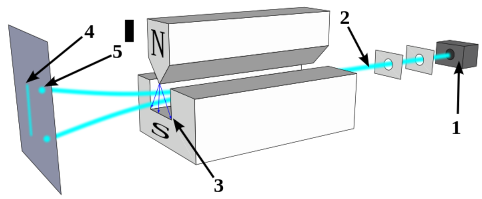

Figure 1.1: Classical expectation versus quantum result of the SG experiment

1.3 Weak Measurement in Stern-Gerlach Experiment

In weak measurement, the wavefunction must not collapse into one of the eigenstates of the measuring operator. Hence, we do not want the output of the device to be well defined eigenstates of spin (up and down). Therefore, we need to find a way to mix the two states when they reach the screen. The distance between the two dots is proportial to the gradient of the magnetic field and the distance of the screen from the inhomogeneous field. By tweaking these two parameters, we will be able to make the two dots overlap over one another. That is, the eigenstates have not been well seperated. Therefore, at the screen, we have a mixed state result.

1.4 Post Selection

In classical mechanics, once the Hamiltonian and the initial values of position and momentum of a particle are known, the particle has a definite path it will follow. Hence, it makes no sense to calculate the probability of a particle passing through a given point in phase space because the answer is either one or zero. Whereas in quantum mechanics, since measurement of a quantity on a wavefunction leads to an eigenvalue with some probability, we can never be sure of the outcome of the measurement. Therefore, one can perform "post selection".

Wilfully forcing the system to be in a specific state after a measurement process is called post selection. This is done by performing many measurements with a myriad outcomes and only selecting the systems with the outcome that is desired, while discarding the rest. For example, in the sequntial Stern-Gerlach experiment, after a measurement of spin, we post select the system to be in spin up or down before making it go through the next Stern-Gerlach apparatus.

Chapter 2

Aharonov-Albert-Vaidman Paper

2.1 The Setup

The probe or the measuring device is defined as a pointer needle that when it "comes across" the system, it measures something.

Immediately after the weak measurement in one direction, we make a strong measurement in the direction perpendicular to both the direction of propagation and the direction weak measurement. Then we select one of the outputs hence selecting a definite final state for the system.

2.2 The Math

2.2.1 Definitions

Let the observable that will be weakly measured: $\hat{A}\,|a_n\rangle = a_n\,|a_n\rangle$
We use the von Neumann Hamiltonian to model the coupling between the system and the probe given by:

$$\hat{H} = -g(t)\hat{q}\hat{A} \tag{2.1}$$

where

- $g(t)$ is a function with compact support near the time of measurement and normalised
- $\hat{q}$ is the canonical variable of the measuring device and $\hat{p}$ is its conjugate momentum

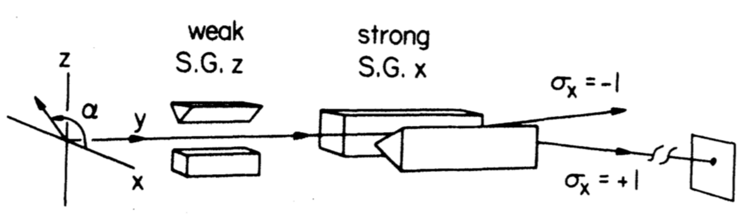

Figure 2.1: Weak measurement followed by strong measurement and post selection. Source: Duck, Sudarshan, Phys.Rev.D (1989)

We also assume that the von Neumann Hamiltonian is the most dominating term in the total Hamiltonian when the measurement is being performed.

Let the initial state of the system be given by $|\Psi_{in}\rangle = \sum_n \alpha_n |a_n\rangle$ and the initial state of the probe be $|\Phi_{in}\rangle$. The probe state can be written in the p and q bases as:

$$|\Phi_{in}\rangle = \int dq \phi_{in}(q) |q\rangle \tag{2.2}$$

$$|\Phi_{in}\rangle = \int dp \widetilde{\phi}_{in}(p) |p\rangle \tag{2.3}$$

We will assume a gaussian distribution in the p-representation centred at 0 with a spread $\Delta p = 1/(2\Delta)$

$$\widetilde{\phi}_{in}(p) = \exp(-\Delta^2 p^2) \tag{2.4}$$

Hence $\Delta q = \Delta$

$$\phi_{in}(q) = \exp(-\frac{q^2}{4\Delta^2}) \tag{2.5}$$

2.2.2 Evolution

The time evolution operator is given by: $exp(-i \int \hat{H} dt)$. Hence the evolved state is given by:

$$\exp(-i \int \hat{H} dt) |\Psi_{in}\rangle |\Phi_{in}\rangle \tag{2.6}$$

Now we perform the following operations:

- Expand of $|\Psi_{in}\rangle$ in terms of $|a_n\rangle$
- Express $|\Phi_{in}\rangle$ in the q representation
- Demand that $\int dt g(t) = 1$

The final expression is given by:

$$\sum_n \alpha_n \int e^{-iqa_n - \frac{q^2}{4\Delta^2}} |a_n\rangle |q\rangle \tag{2.7}$$

To get the expression in p representation, we insert the completeness relation: $\int dp |p\rangle \langle p| = 1$ into the above expression and integrate out the q

$$= \sum_n \alpha_n \int dp \exp(-\Delta^2 (p - a_n)^2) |a_n\rangle |p\rangle \tag{2.8}$$

We then let $\langle p|$ act from the left and take the absolute square, to obtain the probability amplitude:

$$\theta_f(p) = \sum_n |\alpha_n|^2 exp(-2\Delta^2 (p - a_n)^2) \tag{2.9}$$

2.2.3 Post Selection - Exact Method

As we mentioned in the previous chapter, we now post select the system to now be in a specific eigenstate of a strong measurement. We perform a strong measurement in an observable $\hat{B}$ whose commutator with $\hat{A}$ is nonzero. The result of this strong measurement are the eigenvalues of $\hat{B}$. We only pick the states that result in a particular eigenstate $|b\rangle$ of $\hat{B}$. If we define the final state of the system in terms of the eigenstates of $\hat{A}$:

$$|\Psi_f\rangle = |b\rangle = \sum_n \alpha'_n |a_n\rangle \tag{2.10}$$

Then the final state of the probe is:

$$|\Phi_f\rangle = \langle\Psi_f|\exp(-i\int \hat{H}dt)|\Psi_{in}\rangle |\Phi_{in}\rangle \tag{2.11}$$

After performing some elementary mathematics as was shown before, we obtain:

$$|\Phi_f\rangle = \sum_n \alpha_n \alpha'_n \int dp \exp(-\Delta^2(p-a_n)^2) |p\rangle \tag{2.12}$$

which is a summation of gaussians centred at the eigenvalues of $\hat{A}$! Let us take a step back and understand what we have obtained. Our probe here, weakly measures some observable $\hat{A}$ in the p basis. If the measurement was a strong one, we obtain one of the eigenvalues of $\hat{A}$ with probability amplitude given by the norm squared of the coefficient associated with the eigenstate in the expression for the initial state of the system. This is trivial quantum mechanics. But due to the weak measurement procedure, we do not obtain this anymore. Given that the system was post selected to a specific state, the probe points to a some value which is a superpositions of gaussians centered at the eigenvalues of $\hat{A}$. The coefficients of the gaussians centered at a_n not only depend on the initial state, but also on the post selected state! It almost seems like the probe points to a value which depends on what happens to the system later, which seems counter intuitive. There is actually a better way to think about this which will be explained in the upcoming sections.

2.2.4 Post Selection - AAV Method

If we examine the properties of the system throughout its journey after requiring it to be post selected, we are enquiring the conditional probability that given that the final state of the system is fixed, what is the probability that the probe measurement yields some value. This value that the probe measures is called the weak value of the measurable $\hat{A}$. Let us define it mathematically as:

$$A_w = \frac{\langle\Psi_f|\hat{A}|\Psi_{in}\rangle}{\langle\Psi_f|\Psi_{in}\rangle} \tag{2.13}$$

Hence,

$$\begin{aligned} |\Phi_f\rangle &= \langle\Psi_f|exp(iq\hat{A})|\Psi_{in}\rangle |\Phi_{in}\rangle \\ &\approx \langle\Psi_f|1 + iq\hat{A} + ...|\Psi_{in}\rangle |\Phi_{in}\rangle \\ &= \langle\Psi_f|\Psi_{in}\rangle [1 + iqA_w + ...] |\Phi_{in}\rangle \\ &\approx \langle\Psi_f|\Psi_{in}\rangle \textstyle\int dq e^{iqA_w - \frac{q^2}{4\Delta^2}} |q\rangle \end{aligned} \tag{2.14}$$

$$|\Phi_f\rangle \approx \langle\Psi_f|\Psi_{in}\rangle \int dp e^{-\Delta^2(p-A_w)^2} |p\rangle \tag{2.15}$$

which is a single gaussian centred at A_w! Through some approximations, we now have a clue of what the result of the superposition of gaussians would lead to. The probe state points at the weak value of $\hat{A}$ which is

A_w. It is interesting to note that there is no bound to the weak value. It need not be any of the eigenvalues of $\hat{A}$ nor does it have to be close to them. It could be anything. If $\Delta = \Delta q$ is small, then we have a large spread in the p representation, hence the weak value will have a shorter spread, and vice versa.

The approximations that we have made in the above calculations are:

1. $|q^n \langle\Psi_f|\hat{A^n}|\Psi_{in}\rangle \| \ll |\langle\Psi_f|\Psi_{in}\rangle|, n \geq 2$
2. $|q^n \langle\Psi_f|\hat{A^n}|\Psi_{in}\rangle| \ll |q\langle\Psi_f|\Psi_{in}\rangle|, n \geq 2$
3. $|qA_w| \ll 1$

If we consider that q in the measurement is of the order of $\Delta q = \Delta$, then we can replace that in the above expressions to obtain:

1. $\Delta \ll min_{n=2,3,\ldots}|\frac{\langle\Psi_f|\hat{A}|\Psi_{in}\rangle}{\langle\Psi_f|\hat{A^n}|\Psi_{in}\rangle}|^{1/n-1}$
2. $\Delta \ll 1/A_w$

This is the condition for weak measurement.

2.2.5 Comments about Weak Value

Performing one weak measurement and post selection leads to a weak value with a wide standard deviation. In order to obtain the weak value precisely, we have to perform the experiment multiple times. Statistically, the standard deviation decreases as $1/\sqrt{N}$. Hence weak value gaussian will get sharper as more measurements are done. In this way, we can obtain the weak value to an arbitrary precision.

The weak value can also be obtained by performing a single measurement of an ensemble with N systems. In this case, the weak observable will become:

$$A_N = \frac{1}{N}\Sigma A_i \tag{2.16}$$

This will now have additional eigenvalues that are equally spaced, between the original eigenvalues. Instead of performing the measurement on a single system multiple times, we perform a single measurement on an ensemble. The result is exactly the same.

The two different methods are illustrated in figures (2.2) and (2.3)

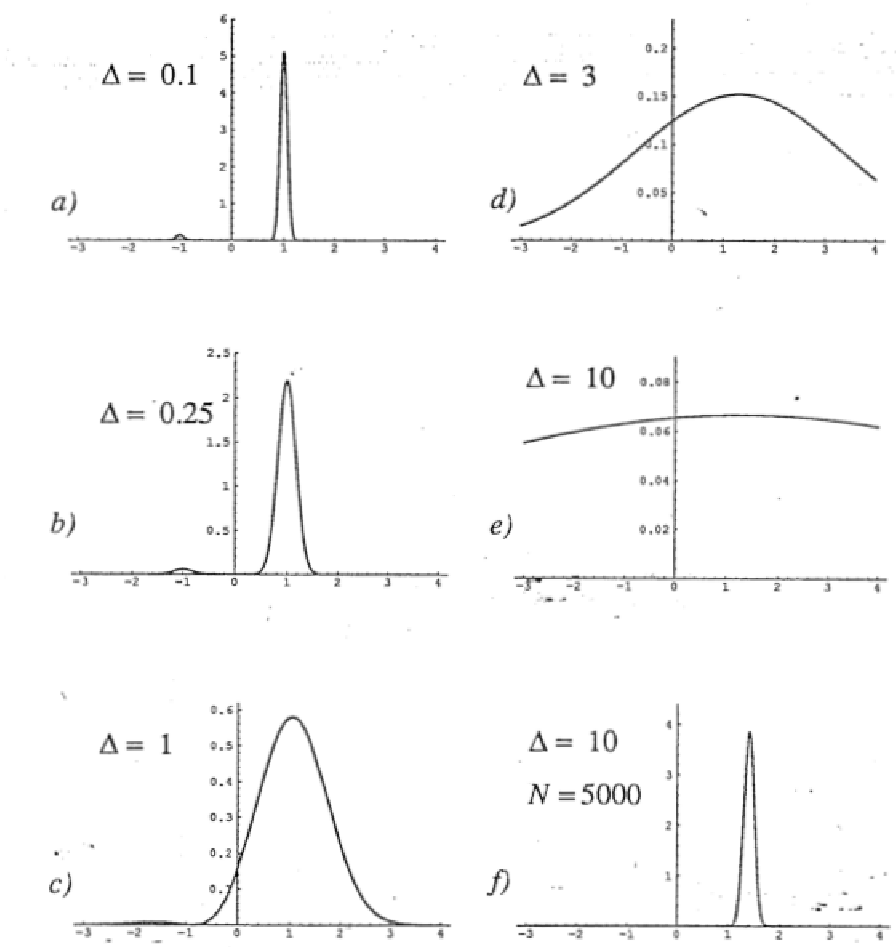

Figure 2.2: Weak value of spin. Source: Vaidman "Weak Measurement" https://arxiv.org/abs/hep-th/9408154

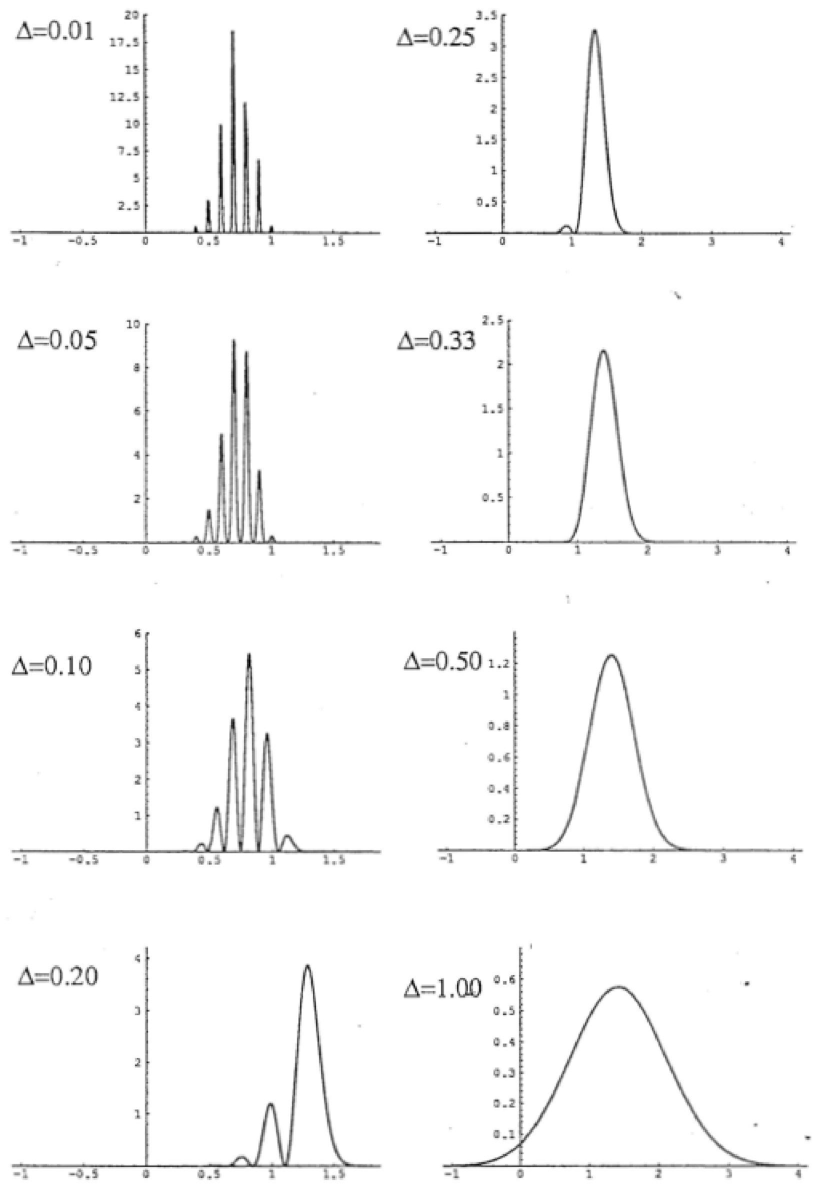

Figure 2.3: Weak value of spin of N particles. Source: Vaidman "Weak Measurement" https://arxiv.org/abs/hep-th/9408154

2.3 Weak Measurement in SG

2.3.1 Setup

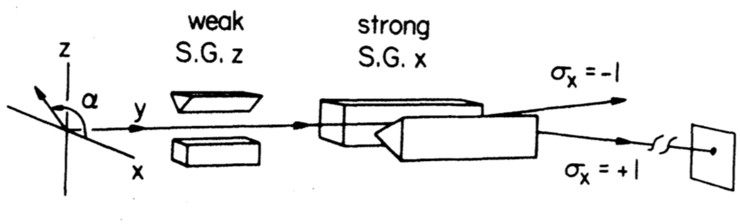

Figure 2.4: Weak measurement followed by strong measurement and post selection. Source: Duck, Sudarshan, Phys.Rev.D (1989)

The figure above explains the paints the whole story in an apparatus we can all visualize. We initially have a system of spin 1/2 particles in the xz plane, oriented at an angle α with respect to the x axis. They then enter an inhomogeneous magnetic field pointing in the z direction. This system measures the z-spin value of the particles. This z-spin measurement is considered to be weak. We then send it through an inhomogeneous magnetic field pointing in the x direction. This is a strong measurement of x-spin. Then we post select the system in the $+1$ eigenstate of σ_x.

The magnetic field in the z direction may not produce any distinct eigenstates of σ_z but it produces a shift in z-momentum p_z. The "weak-ness" of the measurement can be determined by the shift in the z-momentum. Hence we use the value of z-momentum of the spin 1/2 particles as the probe in this case. The system is the spin 1/2 particles in the xz plane while the probe is the z-momentum.

Hence the Hamiltonian is given by:

$$\hat{H} = -\lambda g(t)\hat{z}\hat{\sigma_z} \tag{2.17}$$

where λ is proportional to the gradient of the magnetic field in the z direction. We list the bunch of correlations:

- $\hat{A} = \lambda\hat{\sigma_z}$
- $\hat{q} = \hat{z}$
- $\hat{p} = \hat{p_z}$
- $\Delta z = \Delta$
- $\Delta p_z = 1/(2\Delta)$

The initial state of the system is the $+1$ eigenstate of $(\cos\alpha)\sigma_x + (\sin\alpha)\sigma_z$, expressed in the basis of σ_z as:

$$|\Psi_{in}\rangle = \frac{1}{\sqrt{2}} \begin{Bmatrix} \cos\alpha/2 + \sin\alpha/2 \\ \cos\alpha/2 - \sin\alpha/2 \end{Bmatrix} \tag{2.18}$$

and the final state is the $+1$ eigenstate of σ_x expressed in the σ_z basis as:

$$|\Psi_f\rangle = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{2.19}$$

Hence, we calculate:

$$\begin{aligned} \langle\Psi_f|\Psi_{in}\rangle &= \cos\alpha/2 \\ \langle\Psi_f|\hat{\sigma_z}|\Psi_{in}\rangle &= \sin\alpha/2 \end{aligned} \tag{2.20}$$

$$A_w = (\lambda\sigma_z)_w = \lambda \tan\alpha/2 \tag{2.21}$$

The gaussian functions are obtained as:

$$\phi_{in}(q) = \exp(-\frac{z^2}{4\Delta^2})f(x,y) \tag{2.22}$$

The summation of gaussians that lead to a single gaussian after approxiamtions by AAV, is obtained as:

$$\widetilde{\phi_f}(p) \approx (\cos\alpha/2)\exp(-\Delta^2(p_z - \lambda\tan\alpha/2)^2) \tag{2.23}$$

gaussian in p/λ centred at $\tan\alpha/2$. The square of the above function gives the distribution of p_z in the final beam.

2.3.2 Playing with validity

We need to beware of the condition to be fulfilled for the AAV approximation to be valid.

$$\Delta \ll \lambda^{-1} min[\tan\alpha/2, \cot\alpha/2] \tag{2.24}$$

Hence α must not go too close to π but it can be taken arbitrarily close to π so as obtain a spread so large that the weak value of spin measurement is nowhere near values that make sense. For spin 1/2 particles, a strong spin measurement in the z direction yields either $+1/2$ or $-1/2$. Since the weak value of an observable has no bound, we can show that the result of this measurement can lead to the bizarre value - 100. This was the claim that AAV put out in their paper.

Put $\alpha = \pi - 2\epsilon; \epsilon \ll 1$ and substitute:

The exact result was a superposition of gaussians: $\phi_f(p) = \sum_n \alpha_n \alpha_n^{'*} exp(-\Delta^2(p-a_n)^2)$

Here we have, exactly two eigenvalues.

- $\alpha_1 = \frac{1}{\sqrt{2}}(\cos\alpha/2 + \sin\alpha/2)$
- $\alpha_2 = \frac{1}{\sqrt{2}}(\cos\alpha/2 - \sin\alpha/2)$
- $\alpha_1' = \alpha_2' = 1/\sqrt{2}$
- $a_1 = \lambda$ and $a_2 = -\lambda$

Substituting all of the above with $\alpha = \pi - 2\epsilon$ into the exact solution, we obtain:

$$\phi_f(p) = \frac{1}{2}((1+\epsilon)\exp(-\Delta^2(p-\lambda)^2) - (1-\epsilon)\exp(-\Delta^2(p+\lambda)^2) \tag{2.25}$$

Superposition of 2 gaussians centred at $p = \pm\lambda$ with unequal weights. Whereas when we look at AAV's weak value, the final state was given by:

$$\widetilde{\phi_f}(p) \approx (\cos\alpha/2)\exp(-\Delta^2(p_z - \lambda\tan\alpha/2)^2) \tag{2.26}$$

Put $\alpha = \pi - 2\epsilon; \epsilon \ll 1$ and substitute to obtain a single gaussian:

$$\widetilde{\phi_f}(p) \approx \epsilon\exp(-\Delta^2(p_z - \lambda/\epsilon)^2)$$

We see that the superposed peak has been shifted to the right by an amount λ/ϵ. If ϵ is smaller, then the shift is larger for the same Δ. This result is valid as long as $\lambda\Delta \ll \epsilon \ll 1$. Hence by tweaking the value of ϵ, i.e., getting arbitrarily close to π, we would be able to obtain a weak value as bizarre as 100 for a spin 1/2 particle.

2.4 Optical Analog

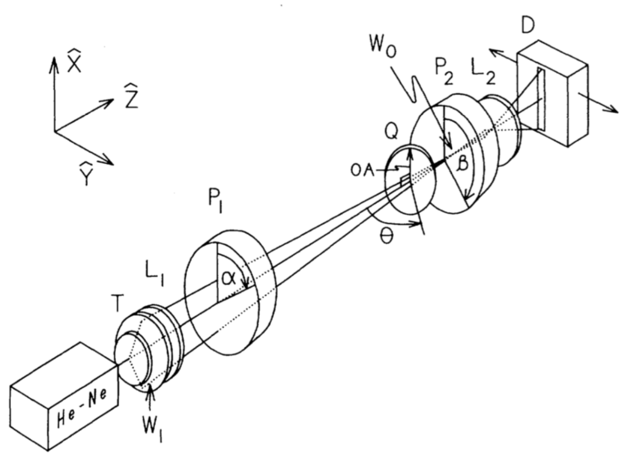

Figure 2.5: Optical version of weak measurement (Source: Ritchie et al. Phys. Rev. Lett. (1990)

We have a lazer beam which is gaussian in the x and y axes and propagating in the z direction. In its path, it encounters a polarizer (P_1) whose optical axis is at an angle α with respect to the x axis. The outgoing light will be polarized along the optical axis. It then comes across a a quarter wave plate (QWP) whose axis is at an angle θ with respect to the z axis. The QWP splits the rays into ordinary and extra-ordinary waves which are displaced from one another in the y axis by an distance $a = a(\theta)$ and have phase difference ϕ. Finally, we post select the system using another polarizer (P_2) whose optical axis is at an angle β with respect to the x axis. At the end of the path, we have an xy detector which measures the intensity of light in each point on the xy plane.

In much the same way as the SG experiment, the probe state is recorded within the beam. The "weak-ness" of the measurement is manifested in the relative displacement between the ordinary and the extra-ordinary waves while the system is the polarization state of light. We start to see some very strong similarities between the SG apparatus and this optical setup. The math will confirm the same.

The electric field vector after the beam passes through P_1 is represented by:

$$\mathbf{E}_i = E_0 G(x) G(y)(\cos(\alpha)\hat{x} + \sin(\alpha)\hat{y}) \tag{2.27}$$

where $G(x)$ denotes gaussian in x centred at 0 with an FWHM of Δ. After passage through the QWP:

$$\mathbf{E}_w = E_0 G(x)[\cos(\alpha) G(y+a) e^{i\phi}\hat{x} + \sin(\alpha) G(y)\hat{y}] \tag{2.28}$$

Finally, post selection by P_2 yields:

$$\mathbf{E}_f = E_0 G(x)[\cos(\alpha)\cos(\beta) G(y+a) e^{i\phi} + \sin(\alpha)\sin(\beta) G(y)](\cos(\beta)\hat{x} + \sin(\beta)\hat{y}) \tag{2.29}$$

Experimentally, we detect the intensity of light instead of the amplitude. Intensity detected at the end is proportional to the square of the electric field at $x = 0$ and is given by:

$$I(y) = I_0[\cos^2(\alpha)\cos^2(\beta) G^2(y+a) + \sin^2(\alpha)\sin^2(\beta) G^2(y) + 2\cos(\phi)\cos(\alpha)\cos(\beta)\sin(\alpha)\sin(\beta) G(y) G(y+a)] \tag{2.30}$$

where I_0 is proportional to $|E_0|^2$. We set $\alpha = \pi/4$.

- If $\beta = \alpha$ then the post selected state is equal to the intial state. Hence the intensity equation is a constructive superposition with the result being a single, unshifted gaussian
- If $\beta = \alpha + \pi/2 + \epsilon$; ($\epsilon \ll 1$), then the post selected state is almost orthogonal to the intial state. This implies a dramatic increase in the weak value. Hence, the intensity equation is a destructive interference with the result being a shifted gaussian with peak at the weak value, $A_w = a_w \approx a\cot(\epsilon)/2$

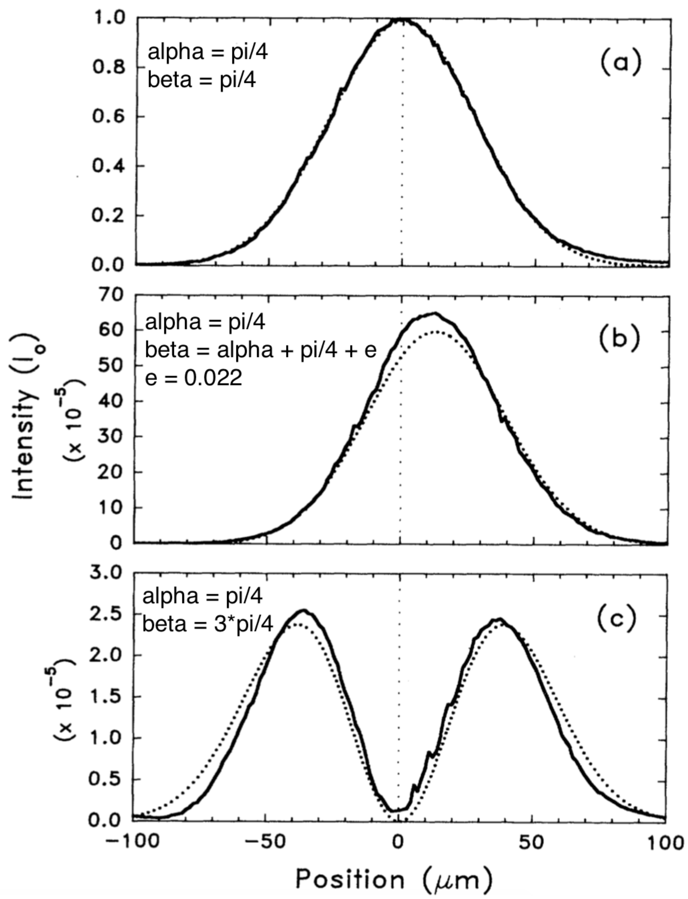

Figure 2.6: Intensity in the detector as a function of y when $a = 0.64\mu$m and $\Delta \approx 55\mu$m (Source: Ritchie et al. Phys. Rev. Lett. (1990))

Chapter 3

Signal to Noise Ratio

3.1 Kedem's Paper

A_w is a complex number in general. Nothing in its definition restricts it to be a real number. What we obtain by repeated measurement here is only the real value of A_w. Real value of A_w affects the measured average value of the weak observable. Whereas the imaginary of A_w:

- Shifts the momentum operator of the probe system.
- Determines the probability of post selection.

We will define later what is meant by the probability of post selection. It is important, however, to note that the imaginary component does not affect the weak value nor the average quantities.

Let us take an observable $\hat{C}$. We define the uncertainity in Q as Δ. If the measurement is performed N times, the standard deviation of Q is $\Delta/\sqrt{N}$. We then define the signal to noise ratio (SNR) as:

$$R = \frac{< Q >_N}{\Delta_N} \tag{3.1}$$

where $< Q >_N$ is the average of value of Q after N measurements. We know that $< Q >_N = N < Q > = Nc$ and $\Delta_N = \sqrt{N}\Delta$. Hence,

$$R = \sqrt{N}\frac{c}{\Delta} \tag{3.2}$$

This is the result for a strong measurement.

Let us look a setup with weak measurement and post selection. Let the system evolve from its initial state $|\Psi\rangle$ to the final state $|\Phi\rangle$. In the AAV regime, $c << \Delta$ and $< Q >_\Phi = Re(C_w)$, hence

$$R = \sqrt{N_\Phi}\frac{Re(C_w)}{\Delta} \tag{3.3}$$

where $N_\Phi = N|\langle\Phi|\Psi\rangle|^2$. The probability of success of post selection is defined as the ratio between N_Φ and N and it indicates the probability of a system reaching the desired final state out of all the final states possible.

If measured in the P basis, according to Jozsa's paper,

$$< P >_\Phi = \Delta^{-2} Im(C_w) \tag{3.4}$$

Here we notice that the average value of momentum depends on the imaginary weak value and also the inverse square of Δ, the spread in Q representaion. This is natural due to the uncertainity principle formed by conjugate pairs of variables. The standard deviation of P is given by $= 1/\sqrt{2}\Delta$. Hence the SNR is:

$$R = \sqrt{N_\Phi}\frac{Im(C_w)}{\Delta} \tag{3.5}$$

Hence if $Re(C_w) = Im(C_w)$ then the signal to noise ratio in both the representations would be the same.

But what if the probe is imperfect? That is, if it has a nonzero average value before even it observes the system?

Suppose the probe is shifted to Q_0. We assume that Q_0 is a gaussian random variable with zero mean and standard deviation equal to Δ_Q. Hence the average and higher moments will be modified as:

$$\begin{aligned} < Q >_\Phi &= Q_0 + Re(C_w) \\ < Q^2 >_\Phi &= \tfrac{\Delta^2}{2} + (Q_0 + Re(C_w))^2 \end{aligned} \tag{3.6}$$

Now we take the average of the above quantities with respect to Q_0 as well:

$$\begin{aligned} \overline{< Q >_\Phi} &= Re(C_w) \\ \overline{< Q^2 >_\Phi} &= \tfrac{\Delta^2}{2} + \tfrac{\Delta_Q^2}{2} + Re(C_w)^2 \end{aligned} \tag{3.7}$$

Hence, the SNR will have the same shift but with larger deviation.

In the same way, we consider a gaussian random variable in the P basis - P_0 with mean zero and deviation Δ_P. By averaging for post selection only, we obtain:

$$\begin{aligned} < P >_\Phi &= P_0 + \Delta^{-2} Im(C_w) \\ < P^2 >_\Phi &= \tfrac{\Delta^{-2}}{2} + (P_0 + \Delta^{-2} Im(C_w))^2 \end{aligned} \tag{3.8}$$

But there is a twist in the tale! Now if we average with respect to P_0 we do not get the same result! Here it is very crucial to remember that the imaginary part of the weak value is responsible for the shift in momentum of the probe. The peak of the P_0 distribution is shifted by $\Delta_P^2 Im(C_w)$. Hence averaging with respect to P_0 is not going to be the same as averaging over Q_0:

$$\begin{aligned} \overline{< P >_\Phi} &= (\Delta^{-2} + \Delta_P^2) Im(C_w) \\ \overline{< P^2 >_\Phi} &= \tfrac{\Delta^{-2}}{2} + \tfrac{\Delta_P^2}{2} + ((\Delta_P^2 + \Delta^{-2}) Im(C_w))^2 \end{aligned} \tag{3.9}$$

Hence the signal to noise ratio is given by:

$$R = \sqrt{N_\Phi} Im(C_w) \sqrt{\Delta^{-2} + \Delta_P^2} \tag{3.10}$$

In this case, both the signal value and the deviation have changed! This equation suggests that if we increase the standard deviation of the intial probe state distribution P_0, we can improve the signal to noise ratio. Another bizarre result! A kind of disordered and messed up probe state is going to give us a better signal when measuring the weak value! Let us see how in the following sections.

3.2 Tamate's Paper

Due to the result above we look for the kind of results we obtain if we use a completely mixed probe state (maximum standard deviation). The time evolution operator for the system and the probe is given by:

$$U(\theta) = exp(-i\theta A \otimes K) \tag{3.11}$$

where θ is a measure of the strength of the coupling between system and the probe. If we are in the AAV regime, we can simplify this as:

$$U_{eff}(\theta) = exp(-i\theta A_w K) \tag{3.12}$$

Let us assume that the probe goes from the initial state σ_i to σ_f. This evolution is captured by the following equation:

$$\sigma_i \rightarrow \sigma_f = P(f/i)U_{eff}(\theta)\sigma_i U_e f f^{\dagger}(\theta) \tag{3.13}$$

Suppose there is an observable M. Its initial and final expectation value can be calculated by using matrix algebra:

$$< M >_{i/f} = \frac{tr(\sigma_{i/f} M)}{tr(\sigma_{i/f})} \tag{3.14}$$

Define the intial and final shift operators as: $\delta_{i/f} M = M - < M >_{i/f}$.

According to Josza's paper, this following result holds good for some observable M.

$$< \delta_i M >_f = i\theta Re(A_w) < [K, M] >_i + \theta Im(A_w) < \{\delta_i K, \delta_i M\} >_i + O(\theta^2) \tag{3.15}$$

The left hand side is the final average value of the intial shift in observable M. The right hand side of the equation is completely expressed in terms of intial averages of the commutator of K and M and the anti commutator of inital shifts of K and M multiplied by real and imaginary weak value respectively up to first order in θ.

Put M = K, we get

$$< \delta_i K >_f = 2\theta Im(A_w) < (\delta_i K)^2 >_i + O(\theta^2) \tag{3.16}$$

This is an important equation because it says that by just evaluating the final average of the initial shift in K - the probe state, we obtain information regarding just the imaginary part of the weak value.

We now make the following assumptions:

- The final variance of K is given by: $< (\delta_f K)^2 >_f = < (\delta_i K)^2 >_i + O(\theta)$
- Probability of successful post selection is given by: $tr\sigma_f = N_\Phi / N + O(\theta)$

By repeating the measurement N times, the post selection succeeds, $Ntr\sigma_f = N_\Phi$ times on average. Using these values, we can now calculate the signal to noise ratio:

$$\begin{aligned} R &= \frac{Ntr(\sigma_f) < \delta_i K >_f}{\sqrt{Ntr(\sigma_f) < (\delta_f K)^2 >_f}} \\ &= 2\theta Im(A_w)\sqrt{N_f < (\delta_i K)^2 >_i} + O(\theta^2) \end{aligned} \tag{3.17}$$

Compare this SNR to the SNR obtained when the instrument was perfect,

$$R = \sqrt{N_\Phi}\frac{Im(C_w)}{\Delta} \qquad \text{(3.5 revisited)}$$

where Δ was the deviation for Q. We see that as long as the standard deviation is maintained to be the same value, the measurement process is not hindered because the signal to noise ratio in the setup depends on the initial probe state standard deviation which is independent of the coherence between the eigenstates of K.

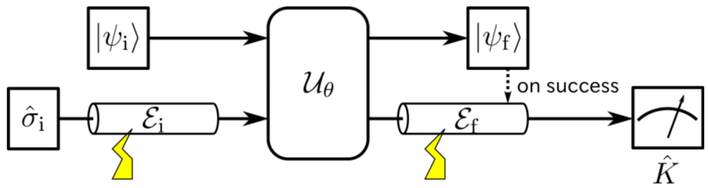

Figure 3.1: Setup for weak measurement with mixed probe states. Noise is introduced before and after probing. http://arxiv.org/abs/1211.4292v1

3.3 Noise tolerance

We assume that the probe is exposed to noise before and after the interaction, expressed by quantum channels $\mathcal{E}_i$ and $\mathcal{E}_f$ as shown in figure 3.1. We assume that the quantum channels satisfy the condition for an arbitrary eigenstate $|k\rangle$ of K:

$$\mathcal{E}(|k\rangle\langle k|) = |k\rangle\langle k| \tag{3.18}$$

This phase noise condition is satisfied if and only if the Kraus representation of the quantum channel is of the form:

$$E_n = \sum_n c_n(k)\,|k\rangle\langle k| \tag{3.19}$$

where $c_n(k)$ are complex numbers satisfying $\sum_n |c_n(k)|^2 = 1$.

Let us define the time evolution of measurement interaction as: $U_\theta(\hat{\rho}) = U(\theta)\hat{\rho}U(\theta)^\dagger$ where $\hat{\rho}$ represents the state of the whole system. If $\mathcal{E}$ is phase noise, then it is assumed to satisfy the following conditions:

$$\begin{aligned} (\mathcal{I}\otimes\mathcal{E})\circ U_\theta &= U_\theta\circ(\mathcal{I}\otimes\mathcal{E}) \\ \langle k|\mathcal{E}(\hat{\sigma})|k\rangle &= \langle k|\hat{\sigma}|k\rangle \end{aligned} \tag{3.20}$$

where $\mathcal{I}$ is the identity channel and $\hat{\sigma}$ is the arbitrary probe state. Let $p'_f(k)$ and $p_f(k)$ denote the final probability distributions with and without noise respectively. If these are equal, we can conclude that the probe is unaffected by the addition of phase noise. Using the properties of phase noise we described above and the fact that composition of phase noises is also a phase noise, we obtain:

$$\begin{aligned} p'_f(k) &= \langle\psi_f|\langle k|(\mathcal{I}\otimes\mathcal{E}_f)\circ U_\theta\circ(\mathcal{I}\otimes\mathcal{E}_i)(|\psi_i\rangle\langle\psi_i|\otimes\hat{\sigma}_i|\psi_f\rangle|k\rangle \\ &= \langle\psi_f|\langle k|\,(\mathcal{I}\otimes\mathcal{E}_f\circ\mathcal{E}_i)\circ U_\theta(|\psi_i\rangle\langle\psi_i|\otimes\hat{\sigma}_i)\,|\psi_f\rangle\,|k\rangle \\ &= \langle\psi_f|\langle k|\,U_\theta(|\psi_i\rangle\langle\psi_i|\otimes\hat{\sigma}_i)\,|\psi_f\rangle\,|k\rangle = p_f(k) \end{aligned} \tag{3.21}$$

The result of the weak measurement is unaffected by the introduction of the phase noises. Even better if we use a completely mixed probe state. Any noise that maps the identiity operator to itself cannot affect the completely mixed probe state.

3.4 Experiment

The setup is given in figure 3.2. The light emitted by a superliminescent laser diode (SLD) is polarized using a Glan laser (GL) polarizer. Then the polarized light is passed thorough a half wave plate (HWP) at 45 degrees with respect to the polarization. Since the coherence time of light is lesser than the differential group delay within the HWP, the emerging light is unpolarized. This unpolarized light enters a Mach-Zender interferometer setup and split into upper and lower paths. The prism in the lower path is fitted on a piezo electric stage with micrometer adjustment. On each arm of the interferometer, there is a HWP. The angle

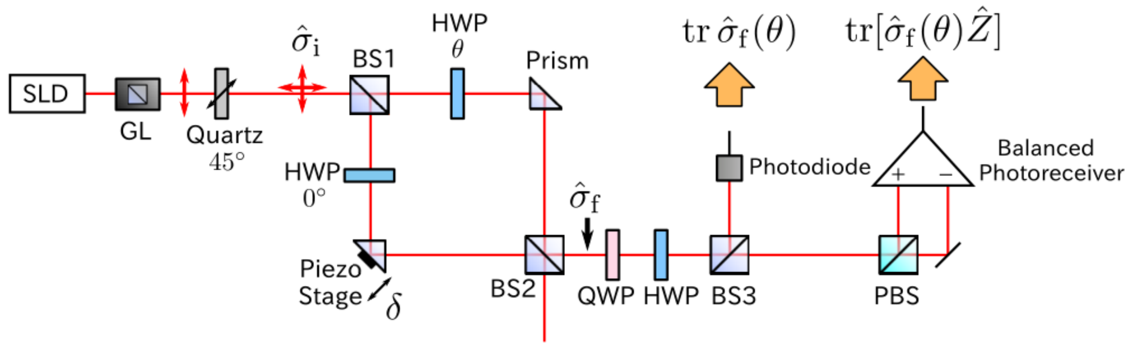

Figure 3.2: http://arxiv.org/abs/1211.4292v1

between the two HWPs, θ, indicates the strength of interaction between the path and the polarization of light. At the output, the power and polarization of light are measured. The QWP and HWP measure the circularly polarized components of light.

The first beam splitter sets the intial state of the system to be:

$$|\psi_i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{3.22}$$

where $|0\rangle$ and $|1\rangle$ are used to denote the upper and lower paths of light. Post selection occurs at the second beam splitter. Due to the relative phase introduced by the moving stage, the post selected state is given by:

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\delta}|1\rangle) \tag{3.23}$$

The relative angle between the two HWP in each arm (θ) serves as the strength of interaction parameter. Let U_{HWP} denote the evolution caused by the HWP. Then for the whole system,

$$\begin{aligned} U(\theta) &= |0\rangle\langle 0| \otimes U_{HWP(\theta)} + |1\rangle\langle 1| \otimes U_{HWP(0)} \\ &= U_{HWP(0)}[|0\rangle\langle 0| \otimes \exp(2i\theta Z) + |1\rangle\langle 1| \otimes I] \end{aligned} \tag{3.24}$$

Z is some observable that distinguishes the two circular polarizations.

By adjusting the basis for polarization, we can eliminate $U_{HWP(0)}$ the effective evolution is:

$$U(\theta) = exp(2i\theta P_0 Z) \tag{3.25}$$

where P_0 is the projection operator to the upper path $(|0\rangle\langle 0|)$.

Now we can just substitute values back into the equations we derived by putting P_0 as A and Z as K. From the weak value formula, we obtain

$$Im(< P_0 >_w) = \frac{1}{2}\tan(\delta/2) \tag{3.26}$$

$$\begin{aligned} tr(\sigma_f(\theta)) &= \tfrac{1}{2}(1 + \cos\delta\cos 2\theta) \\ tr(\sigma_f(\theta)Z) &= -\tfrac{1}{2}\sin\delta\sin 2\theta \end{aligned} \tag{3.27}$$

Using these values,

$$Im(< P_0 >_w) = -\frac{1}{4}\frac{d}{d\theta}\bigg|_{\theta=0} \frac{tr(\sigma_f(\theta)Z)}{tr(\sigma_f(\theta))} \tag{3.28}$$

This is experimentally obtained by plotting and finding the slope near $\theta = 0$

3.5 Visibility

The imaginary weak value was found to be 2.26 which is not large. This is because the post selection was not complete. We define the parameter called visibility (V) which when 1 indicates fully post selected state and when 0 indicates identity (zero post selection probability). Let the actual post selected state be: ρ_f
In the case of no interaction: $\rho_f = V \left|\psi\right\rangle \left\langle\psi\right| + (1 - V)\frac{I}{2}$

Hence the new weak value in terms of V:

$$Im(< P_0 >_w) = \frac{V \sin\delta}{2(1 + V\cos\delta)} \tag{3.29}$$

when V = 1, we get back the original case.

The results obtained by Tamate et al. is shown in figure 3.3. We can see that if the visibility factor is equal to one, there is a divergence at the relative phase value equal to π. A visibility factor of 0.977 seems to fit their experimental data well.

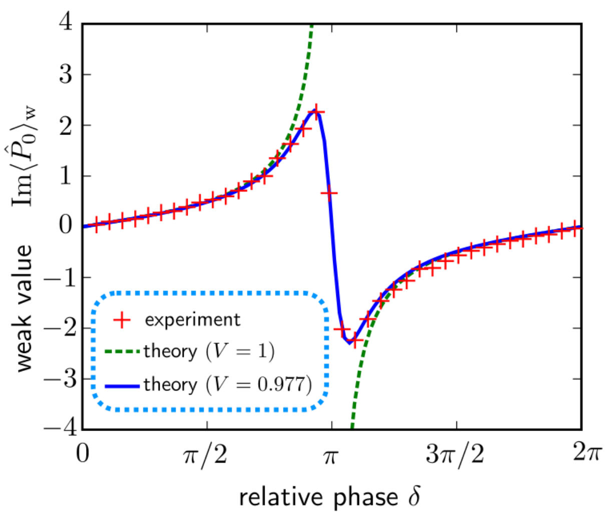

Figure 3.3: http://arxiv.org/abs/1211.4292v1

Chapter 4

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