Summer Project Report

TNSA and Mathematical Modelling of Plasma

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Introduction

The paper[4] investigates the screening effect caused in plasma for thermonuclear fusion reactions occurring in stars. Since plasma with the exact physical conditions of a core of a star cannot be replicated in the laboratory, they investigate the plasma that could be generated in a lab using high intensity lasers. The screening effect results obtained here can at least give a glimpse of what might happen in the dense core of stars.

They go about this by proposing an experiment that can be performed in the ELI-NP laboratory soon to be open for experimental use in Romania. The rate of the following fusion reaction was being studied in the proposed experiment.

$$^{13}\mathrm{C} + {}^{4}\mathrm{He} \longrightarrow {}^{16}\mathrm{O} + {}^{1}\mathrm{n}$$

The experiment starts with the expulsion of ^{13}C ions from the first solid target via the TNSA mechanism. These ions then meet the gas He target or the plasma He target (produced by a second laser beam) and the reaction products are detected on the other side.

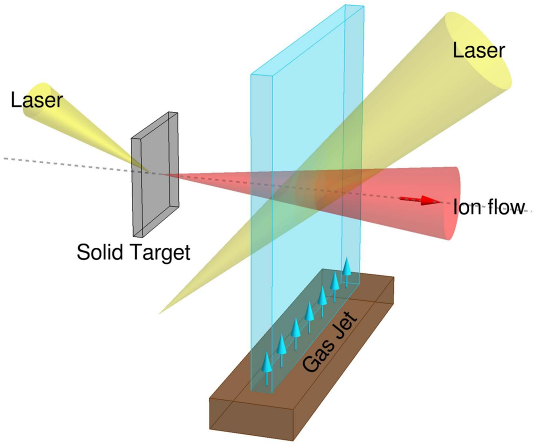

Figure 1: Schematic of the proposed experiment

The results obtained in the paper can be divided into 3 parts:

- Number of ^{13}C ions expelled from the rear side of the target by TNSA mechanism per unit energy as a function of energy and how it changes with change in laser characteristics that produced it.
- Rate of fusion reaction as a function of temperature and how it is affected by altering the density of helium and the charge number of the carbon ion.
- Number of neutron events per laser pulse per unit energy as a function of energy and how it is affected by temperature.

The results are numerical simulations based on the TNSA scheme in laser-target interaction which uses the fluid model of plasma expansion into vacuum proposed by Mora (2003).

Results from the Mora (2003) and Fuchs et al. (2006) papers

Mora proposed a one dimensional, isothermal fluid model to to study the charge separation effects in collision-less plasma. Based on the results obtained from the model ,we can determine the maximum ion energy and the ion energy spectrum. Here are the relevant formulae from the two papers and the references therein. A description of the variables are given in a table separately. SI units with $k_B = 1$ are adopted.

The number of accelerated ions per unit energy is given by:

$$\frac{dN}{dE} = \frac{n_{\mathrm{i0}} c_{\mathrm{s}} t_{\mathrm{acc}} S_{\mathrm{sheath}}}{\sqrt{2E\mathcal{E}_0}} exp(-\sqrt{2E/\mathcal{E}_0}) \tag{1}$$

$n_{\mathrm{i0}} = n_{\mathrm{e0}}/Z_{\mathrm{i}}$ (quasi-neutral plasma); $n_{\mathrm{e0}} = N_{\mathrm{e}}/(c\tau_{\mathrm{laser}} S_{\mathrm{sheath}})$; $c_{\mathrm{s}} = \sqrt{Z_{\mathrm{i}} T_{\mathrm{e}}/m_{\mathrm{i}}}$

$$T_{\mathrm{e}} = m_{\mathrm{e}} c^2 [\sqrt{1 + I\lambda^2_{\mu m}/1.37 \times 10^{18}} - 1] \tag{2}$$

$\mathrm{t_{acc}} = 1.3\tau_{\mathrm{laser}}$; $\mathcal{E}_0 = \mathrm{Z_i T_e}$

$$N_{\mathrm{e}} = f_{\mathrm{abs}} E_{\mathrm{laser}}/T_{\mathrm{e}} \tag{3}$$

where,

$$f_{\mathrm{abs}} = 1.2 \times 10^{-15} I^{0.74} \tag{4}$$

with a maximum at 0.5

$$S_{\mathrm{sheath}} = \pi(r_0 + d_{\mathrm{t}} tan\theta)^2 \tag{5}$$

For a Gaussian pulse, $r_0 = 1.18w/2$

$$I = \frac{P}{\pi w^2/2}$$

and,

$$E_{\mathrm{laser}} = P\tau_{\mathrm{laser}}/0.94$$

The maximum cut off energy that can be gained by accelerating ions is:

$$E_{\mathrm{max}} = 2\mathcal{E}_0[ln(t_{\mathrm{p}} + \sqrt{{t_{\mathrm{p}}}^2 + 1})]^2 \tag{6}$$

$t_{\mathrm{p}} = \omega_{\mathrm{pi}} t_{\mathrm{acc}}/\sqrt{2Exp}$;

$$\omega_{\mathrm{pi}} = \sqrt{n_{\mathrm{e0}} Z_{\mathrm{i}} e^2/(m_{\mathrm{i}}\epsilon_0)}$$

Finally, the electric field at the ion front as predicted by Mora is:

$$E_{\mathrm{front,0}} = \sqrt{2/Exp}(n_{\mathrm{e0}} T_{\mathrm{e}}/\epsilon_0)^{1/2} \tag{7}$$

This is useful to determine the nature of the carbon ion expelled from the rear surface of the target.

Table 1: Description of Variables

Variable	Description
N	Number of carbon 13 ions expelled from the target rear
E	Energy of the carbon 13 ions
$\mathrm{n_{i0}}$	Initial density of ions (unperturbed plasma)
$\mathrm{n_{e0}}$	Initial density of electrons (unperturbed plasma)
$\mathrm{Z_i}$	Ion charge state
$\mathrm{c_s}$	Ion sound speed
$\mathrm{m_i}$	Ion mass (in SI units)
$\mathrm{m_e}$	Mass of an electron in SI units
$\mathrm{N_e}$	Total number of electrons accelerated to the target
c	Velocity of light
$\mathrm{t_{acc}}$	Effective acceleration time ($1.3\tau_{\mathrm{laser}}$)
$\mathrm{S_{sheath}}$	Surface over which the electrons spread on the target
$\mathrm{r_0}$	Initial radius of the zone over which the electrons are accelerated at the target surface
w	FWHM of the Gaussian laser pulse
P	Peak power of the laser
I	Peak intensity of the laser (W/cm^2)
τ_{laser}	Laser pulse duration
ω_{pi}	Ion plasma frequency
$\mathrm{t_p}$	Normalised acceleration time
e	Unit charge
ϵ_0	Permittivity in free space
$\mathcal{E}_0$	$\mathrm{Z_i T_e}$
$\mathrm{T_e}$	Electron temperature

Table 1 continued from previous page

$\lambda_{\mu m}$	Laser wavelength in micro meters (0.8)
$\mathrm{f_{abs}}$	Fraction of laser light absorbed into plasma as fast electrons
$\mathrm{E_{laser}}$	Laser energy
$\mathrm{E_{front,0}}$	Initial electric field at the ion front
Exp	exp(1)
$\mathrm{d_t}$	Thickness of the target
θ	Half angle divergence of hot electrons inside the target (25 degrees)

Summary of Results

Two different laser configurations were used:

1. $I = 10^{20}$ W/cm^2, $P = 10$ PW and $\tau_{\text{laser}} = 25$ fs
2. $I = 10^{19}$ W/cm^2, $P = 1$ PW and $\tau_{\text{laser}} = 250$ fs

The $\mathrm{E_{front,0}}$ values for each of the cases is: 1.39×10^{13} V/m and 2.30×10^{12} V/m respectively. Comparing these values with the threshold electric field for the creation of n+ charge state of carbon (Hegelich et al. 2002), we can see that, in the first case, carbon will have +6 ionic state while in the second case, it will not cross +4. $\mathrm{Z_i}$ for each of the cases was changed accordingly.

Table 2: Threshold electric field for the creation of n+ ionic state of carbon (Hegelich et al. (2002))

n	Threshold Electric Field (V/m)
1	2.2×10^{10}
2	5.2×10^{10}
3	1.3×10^{11}
4	1.8×10^{11}
5	5.3×10^{12}
6	7.0×10^{12}

The empirical fraction $\mathrm{f_{abs}}$ crosses its threshold value of 0.5 in the first case, hence its value is reset to the threshold value.

The maximum cut-off energy that can be gained by the accelerating ions are ($\mathrm{E_{max}}$): 22.19 MeV and 14.80 MeV respectively.

A graph of the number of ^{13}C ions accelerated per unit energy as a function of energy was plotted for the two cases (mysim.py).

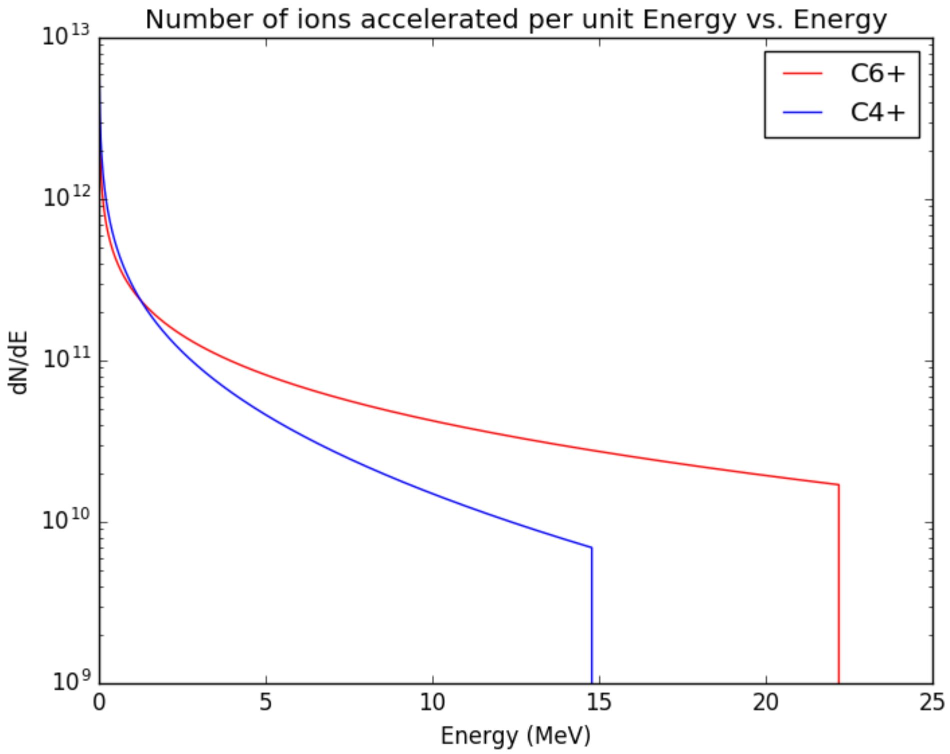

Figure 2: Number of TNSA accelerated carbon ions per unit energy. Thickness of the target is 5 μm

Why the new model?

The model described in Mora (2003) paper, is an isothermal, semi-infinite, one dimensional plasma model. The model was proposed to study the charge separation effects in a collisionless plasma. While the model is able to give good estimates on the number of ions emitted per unit energy from a target rear surface, it assumes that the electron temperature remains constant. This assumption is plausible during a laser pulse, it is violated for time scales beyond that, since electrons progressively lose their energy to the ions in the plasma. Another drawback of the old model is that it does not describe charge separation effect and the structure of the ion front which is crucial in determining the energy of the accelerated ions.

The New Model

The new adiabatic model[2] does not assume the temperature of the electrons to be constant and takes into account the cooling of electrons by transfer of energy to the ions. The model deals with the collision-less expansion into vacuum of a thin foil of initial width L, as opposed to the semi-infinite model before.

At t=0, the ions are cold and at rest with an initial density $n_{i0} = n_{e0}/Z_i$ within a one dimensional box of length L (centered at the origin). The equations of the model are the same as the model before (Boltzmann equation for n_e, Poisson equation for the potential, equations of continuity and motion for n_i and v_i) except the boundary conditions are different for the left part of the box, E(x=0)=0 and v_i(x=0)=0 for all time t. Also the electron temperature is a function of time governed by the equation of conservation of energy:

$$dU_{\mathrm{e}}/dt = -dU_{\mathrm{ions}}/dt - dU_{\mathrm{field}}/dt \tag{8}$$

U_{ions} is the kinetic energy of the ions, U_{field} is the electrostatic energy of the electric field and U_e is the thermal energy of electrons (all defined per unit surface).

$$U_{\mathrm{e}} = g(\theta) N_{\mathrm{e}} T_{\mathrm{e}} \tag{9}$$

where $N_e = n_{e0}L$ is the total number of electrons and g(θ) is a function of $\theta = T_e/m_ec^2$. We can think of the function g like a fractional co-efficient which is the ratio between the actual thermal energy of the electrons and the average thermal energy of the electrons. g=1/2 in the classical limit ($\theta = 0$) and g=1 in the ultra-relativistic limit ($\theta = \infty$).

Energy of the electron can also be calculated by the work done on the electron fluid by the electric field:

$$dU_{\mathrm{e}}/dt = -e\int_{-\infty}^{\infty} E n_{\mathrm{e}} v_{\mathrm{e}} dx = -T_{\mathrm{e}}\int_{-\infty}^{\infty} n_{\mathrm{e}} \partial v_{\mathrm{e}}/\partial x dx = e\int_{-\infty}^{\infty} \Phi \partial n_{\mathrm{e}}/\partial t dx \tag{10}$$

The characteristic expansion time of the foil is taken to be $t_L = L/2c_{s0}$ which is the time taken by the electrons to reach the center of the foil if their temperature is held constant. For $t \ll t_L$ the expansion is similar to the old model. For t close to t_L, electron cooling progressively occurs. Finally for $t \gg t_L$ the electron cooling process is fully effective and the velocity becomes progressively zero, with v(x,t) $\simeq$ x/t. Also, the density profile n_e(x,t) is inversely proportional to time.

Computation

Continuity equations for ion density and ion velocity,

$$\frac{\partial n_i}{\partial t} + v_i \frac{\partial n_i}{\partial x} = -n_i \frac{\partial v_i}{\partial x} \tag{11}$$

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} = -\frac{Ze}{m_i} \frac{\partial \Phi}{\partial x} \tag{12}$$

The Poisson equation,

$$\epsilon_0 \frac{\partial^2 \Phi}{\partial x^2} = e(n_e - Zn_i) \tag{13}$$

Electron density relation,

$$n_e = n_{e0} exp(\frac{e\Phi}{T_e}) \tag{14}$$

and the differential equation for electron temperature,

$$\frac{dU_e}{dt} = n_{e0} L g(\theta) \frac{dT_e}{dt} = e \int_{-\infty}^{\infty} \Phi \partial n_{\mathrm{e}} / \partial t dx \tag{15}$$

With initial conditions: $\Phi(x = 0, t) = 0$, $v_i(x = 0, t) = 0$, $n_i(x, 0) = n_{e0}/Z$ if $|x| < L$, otherwise $n_i(x, 0) = 0$, $T_e(t = 0) = T_{e0}$, $v_i(-L, t) = 0$ and $E(-L, t) = 0$ where $E = -\partial\Phi/\partial x$ For the continuity equations, we can make use of the self similar variable[3]: $\xi = x/c_{s0}t$ and modify the equations into ordinary differential equations that can be solved easier.

$$(u_i - \xi) \frac{dlnn_i}{d\xi} = -\frac{du_i}{d\xi} \tag{16}$$

$$(u_i - \xi) \frac{du_i}{d\xi} = -\frac{d\phi}{d\xi} \tag{17}$$

where $u_i = v_i/c_{s0}$ and $\phi = e\Phi/T_e$ The self similar solution for ϕ is: $\phi = -1 - \xi$. The solutions of these equations (backup_nsssim.py) has been plotted in figures 3 and 4 (Electron temperature held constant):

A way to solve these equations (11-15) is that we can use the initial self-similar assumption for the potential and plug it in the continuity equations (eqns: 11 and 12) to obtain the ion density and velocity, then use the same potential to find the electron density (eqn:14) with electron temperature being T_{e0}. Then use the electron and ion density to solve the Poisson equation (eqn:13) to obtain the new potential and use the new potential to find out the electron temperature from eqn:15. Plug the new potential into the continuity equation and the cycle goes on until a stable, converging solution is obtained.

I tried this with my code (3nsssim.py) but I got stability issues with the result. There was a run away effect after 10 steps or so. I am not sure if the problem lies with the

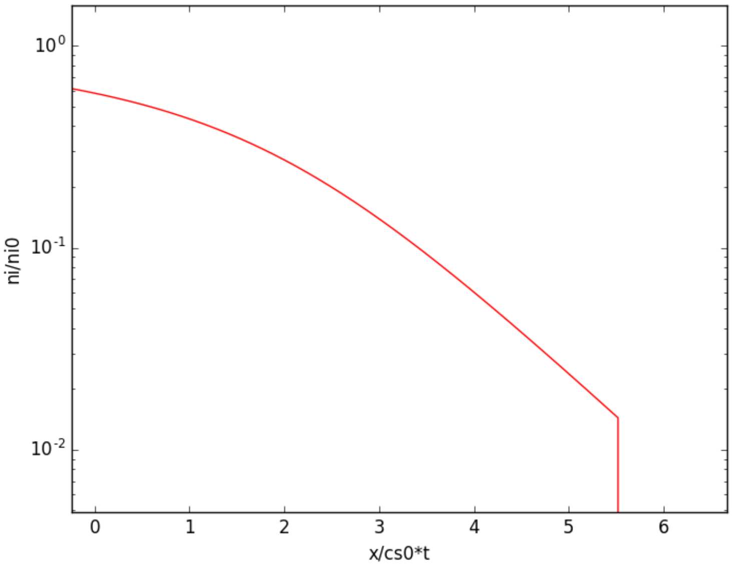

Figure 3: Ion density as a function of ξ for a proton at $\omega_{pi}t = 50, L = 20\lambda_0, T_{e0} = 1$ MeV

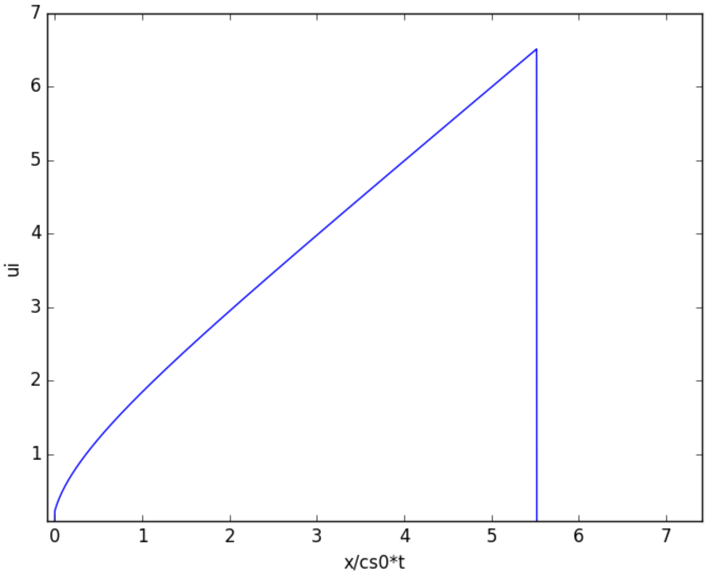

Figure 4: Velocity of ion as a function of ξ for a proton at $\omega_{pi}t = 50, L = 20\lambda_0, T_{e0} = 1$ MeV

integration in the differential equation involving electron temperature or if the equations are just not converging. I am yet to obtain an acceptable result. I have also tried using mathematica but in vain.

Remarks

The 2003 Mora paper talks about using a Lagrangian code to solve the 5 equations in question and the method used is apparently the same as the one used in True,Albritton,Williams (1979) paper. In that paper, a three fluid model is used (ions, hot and cold electrons). A look into the paper did not give me a satisfactory computational algorithm to solve the equations. If there was more information available on the Lagrangian code, then replicating solution will be a lot easier. Otherwise, converging method looks to be the only way to go.

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