Bohm Aharonov Effect

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1 Introduction

Looks like electromagnetic potentials are not required since they are gauge invariant anyway and they may not carry any physical meaning. Fundamental equations of motions are expressed in terms of fields alone. But we are wrong.

2 Experiment

Faraday cage with time varying potential $V(t)$ due to an external generator. Hamiltonian $H = H_0 + V(t)$ where H_0 is the Hamiltonian of the system when the generator is not working. If ψ_0 is the eigenfunction of the Hamiltonian H_0 then the eigenfunction of H is: $\psi = \psi_0 e^{-iS/\hbar}$ where $S = \int V(t)dt$. This follows from a simple calculation:

$$i\hbar\frac{\partial\psi}{\partial t} = e^{-iS/\hbar}[i\hbar\frac{\partial\psi_0}{\partial t} + \psi_0\frac{\partial S}{\partial t}] = [H_0 + V(t)]\psi = H\psi \tag{1}$$

The difference is just a phase factor.

Now let's go for a more complex experiment. Suppose there is a coherent beam of electrons that is split into two beams. Each path consists of a long cylindrical metal tube through with the electron beam must enter. The electron beams are made to coherently interfere at the other end of the tube. Instead of sending a continuous electron beam, we have electrical shutters, that allows electron beam to pass as pulses or more like a wave-packet. The wave-packet is large compared to the Debye wavelength λ but short compared to the dimensions of the cylindrical tube.

The potential in each tube is in such a way that in region 1, the potential is zero until the electron beam is well within the tube. In region 2, the potential rises (although differently in each tube) and then falls back to zero in region 3 (near the edge of the tube). The aim of this set up is so that the electron passes through a time varying potential where there is no field (around the middle of the conducting cylindrical tube).

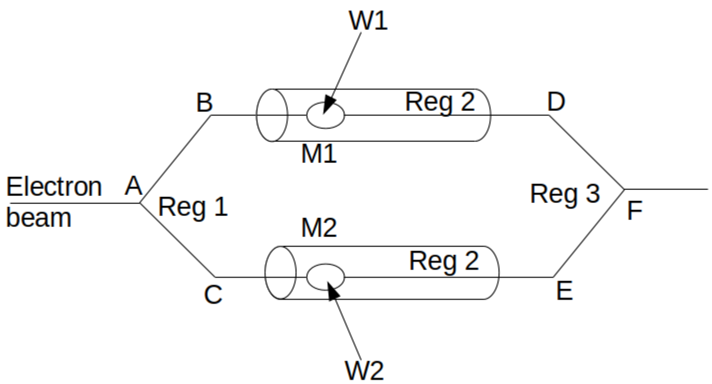

Figure 1: Schematic experiment to determine interference of time dependent scalar potential. A,B,C,D,E are devices used to split/divert beams, F is the interference point. W1,W2 are wave-packets. M1,M2 are cylindrical metal tubes.

Now if we take the eigenfunction of the total Hamiltonian of this system (considering ψ_1^0 and ψ_2^0 to be eigenfunctions when the generator of the time-varying potential is not working)

$$\psi = \psi_1 + \psi_2 \tag{2}$$

Since this problem is quite similar to the first one with the Faraday cage, it is not surprising that the answer is similar.

$$\psi = \psi_1^0 e^{-iS_1/\hbar} + \psi_2^0 e^{-iS_2/\hbar} = (\psi_1^0 e^{-i(S_1-S_2)/\hbar} + \psi_2^0) e^{-iS_2/\hbar} \tag{3}$$

where, $S_1 = \int V_1(t)dt = e\int \phi_1 dt$ and $S_2 = e\int \phi_2 dt$
From this, we can conclude that the interference that takes place at the point F depends on the phase difference: $(S_1 - S_2)/\hbar$. Hence we are able to see a physical effect nothing to do with field and everything to do with potential.
Let us take relativity into account. The 4-potential is expressed as:

$$A_\mu = (\phi, \vec{A})$$

where $\vec{A}$ is the vector potential. Since the eigenfunction above is covariant, we can look for similar results using the vector potential.
The phase difference can be expressed as a closed integral as follows:

$$\frac{e}{\hbar}\oint(\phi dt - \frac{\vec{A}\cdot\vec{dx}}{c}) \tag{4}$$

where the path of integration is over any closed loop in space-time.
Now let us consider a path in space only. In that case, the equation above suggests that the phase difference depends on the vector potential. The phase difference now is:

$$\frac{\triangle S}{\hbar} = -\frac{e}{c\hbar}\oint \vec{A}\cdot d\vec{x} \tag{5}$$

But the value of the closed integral is equal to the total magnetic flux inside the circuit (Φ)

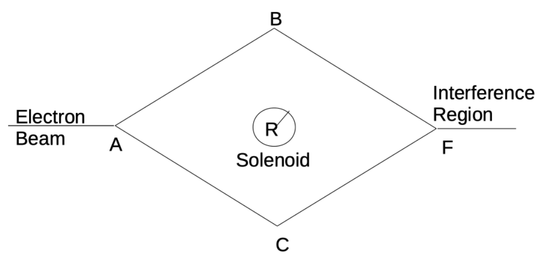

Figure 2: Schematic experiment to demonstrate interference with time-dependent vector potential.

Let us now try to find an experimental situation which will correspond to this situation. If we place a closely wound cylindrical solenoid of radius R centered at the origin and axis in the z direction, we create a magnetic field ($\vec{H}$) which is essentially concentrated within the solenoid. However the vector potential ($\vec{A}$) cannot be zero everywhere outside the solenoid because the total flux through every circuit containing the origin is equal to a constant.

$$\Phi_0 = \int \vec{H}\cdot d\vec{s} = \oint \vec{A}\cdot d\vec{x} \tag{6}$$

As shown in the figure, we have a beam of electrons (this time no wave packets required). The beam is split into two parts, each going on opposing sides of the solenoid kept at the center but avoiding it. The solenoid can be shielded from the beam by keeping a thin metal plate which casts a shadow. The beams are made to interfere again at F.
The Hamiltonian is given by:

$$H = \frac{[\vec{P} - (e/c)\vec{A}]^2}{2m} \tag{7}$$

The equation $\vec{H} = \nabla \times \vec{A}$ works for singly connected regions and one can obtain a solution by taking $\psi = \psi_0 e^{iS/\hbar}$, where ψ_0 is the solution when $\vec{A} = 0$. But in the experiment discussed above, we have a multiply connected region as shown below. Hence ψ as given above is a non-single valued function and therefore

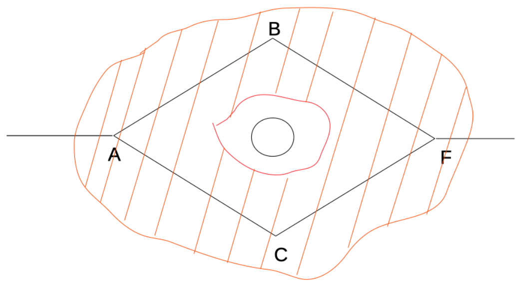

Figure 3: Region outside the solenoid where we are trying to find the solution to the Hamiltonian

not a permissible solution to the Schodinger's equation. But nonetheless, we can split the above region into two singly connected upper and lower regions and find individual solutions for the top (ψ_1, S_1) and bottom (ψ_2, S_2) beams and find the phase difference between the two beams in that way. The phase difference is given by

$$(S_1 - S_2)/\hbar = e/(\hbar c) \oint \vec{A} \cdot \vec{dx} = (e/\hbar c)\Phi_0 \tag{8}$$

This effect persists even though there is no magnetic field in the regions where the beams pass. The result will not be changed even if we surround the solenoid by a potential barrier that reflects the electrons perfectly.
The effect of the vector potential is to produce a shift in the relative phase of the wave function. When the experiment is carried out practically, one observes an interference pattern when the solenoid is turned on or off, since the electron beams interfere either way. Instead, we have to slowly increase the total magnetic flux from zero in order to see the shift in the interference pattern at F. This result would verify the predicted phenomena.
When the magnetic flux is altered, there will be an induced electric field outside the solenoid as predicted by Lenz's law. But these effects can be made negligible since the effect only exists for a very short period of time such that only a small part of the beam would be affected by it.

3 Exact Solution

Aharonov and Bohm, in their paper, find the exact solution to the problem of scattering of an electron beam by a magnetic field in the limit where the magnetic field region tends to zero radius, while the total flux remains fixed. This corresponds to the setup in the second case of the experiment described above but the region of space is not split into two parts instead, the problem is dealt with in a multiply connected region.
Briefly, the method of solving consists of expressing the wave equation in cylindrical polar coordinates and solving for it. The wave equation in cylindrical polar coordinates is:

$$[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}(\frac{\partial}{\partial \theta} + i\alpha)^2 + k^2]\psi = 0 \tag{9}$$

where $\vec{k}$ is the wave vector of the incident particle and $\alpha = -\frac{e\phi}{ch}$ choosing the gauge in which $A_r = 0$ and $A_\theta = \Phi/2\pi r$
The solution takes the form of an infinite sum of Bessel functions with coefficients that one needs to find using boundary conditions and the constant flux condition. The final answer comes out as:

$$\psi \rightarrow e^{-i(\alpha\theta + r' cos\theta)} + \frac{e^{ir'}}{(2\pi i r')^{1/2}} sin\pi\alpha \frac{e^{-i\theta/2}}{cos(\theta/2)} \tag{10}$$

At the line $\theta = \pi$, this solution would show that the second term would combine with the first to make a single-valued wave function despite the non-single valued character of the two parts, in the neighbourhood of $\theta = \pi$.
In the experiment described above, the diffraction effects by the scattering wave have been neglected which is represented by the second term of the above solution. Here, we see that the phase of the wave function has a different value depending on whether we approach the line $\theta = \pm\pi$ from the positive or negative angles (i.e. from the upper or the lower side). This confirms the conclusions from the approximate treatement as given in the previous section.

4 Significance of the Result

From the above experiment and the calculation from the paper, one can conclude that in a field-free multiply connected region, the physical properties of the system depend on potentials which depend on the invariant quantity which is the total flux (Φ).
It is true that, total flux can be expressed in terms of the magnetic field inside the circuit. However, according to relativistic notions, fields act only locally and since electrons cannot reach the regions in which the fields are acting, we cannot attribute the change in phase due to the fields themselves.
In classical mechanics, one can describe the equations of motion in terms of field variables and hence potentials were thought of as mathematical tools with no

physical meaning and fields are the ones that give us values that have physical meaning.
In quantum mechanics, we deal with Schrodinger's equation which includes both field and potentials. The Lorentz force on a particle is derived from a classical approximation and appears nowhere in the fundamental theory. Hence one may lean toward concluding that potentials are fundamental in quantum theory and fields are derived by differentiating them.
But Gauge invariance tells us that if the potentials are subjected to a transformation by a gradient of a scalar continuous function, then all the physical quantities are left unchanged. Hence the same physical behaviour is obtained by two different potentials related by a gradient. Therefore, it was concluded that potentials cannot have any physical meaning and are only used as mathematical tools to calculate the fields.
The above treatment and results, suggest that a new non-local theory must be formulated, or there must be a new way to define and interpret potentials i.e. potentials must be attributed to having some physical meaning since they bring about a change in physical observation when there is no field present. Therefore, we must be able to differentiate between two quantum systems with potentials that differ by a gauge transformation.

5 Reference

Y. Aharonov and D.Bohm "Significance of Electromagnetic Potentials in the Quantum Theory" Phys. Rev. Vol. 115 (1959)